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THEORY OF PROPELLERS

by

Hans Reissner

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Providence, R. I.

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THEORY OF PROPELLERS

Two approaches to propeller theory are possible, namely, the purely dynamical reaction theory for any kind of actuator which produces a slip stream (or jet), and the blade theory which treats the screw-propeller proper as a system of airfoils rotating about and advancing along the axis of the driving shaft. The first theory gives the fundamental characteristics and the upper limits of thrust versus torque, air-speed, and power not only for propellers but also for wind motors and ventilators (fans); the second theory determines the distribution of diameter and blade chords best adapted for a required performance.

Both theories as far as they overlap agree with each other and supplement each other in special points.

<u>The Reaction Theory</u>.

Assume any mechanism (actuator) which creates in its plane of action (propeller disc) a sheet of pressure jump (generalization of the pressure difference between the suction and pressure surface of an airfoil). The pressure jump will produce an acceleration of the fluid and by it a so-called propeller jet (or slip stream). See Fig. 1.

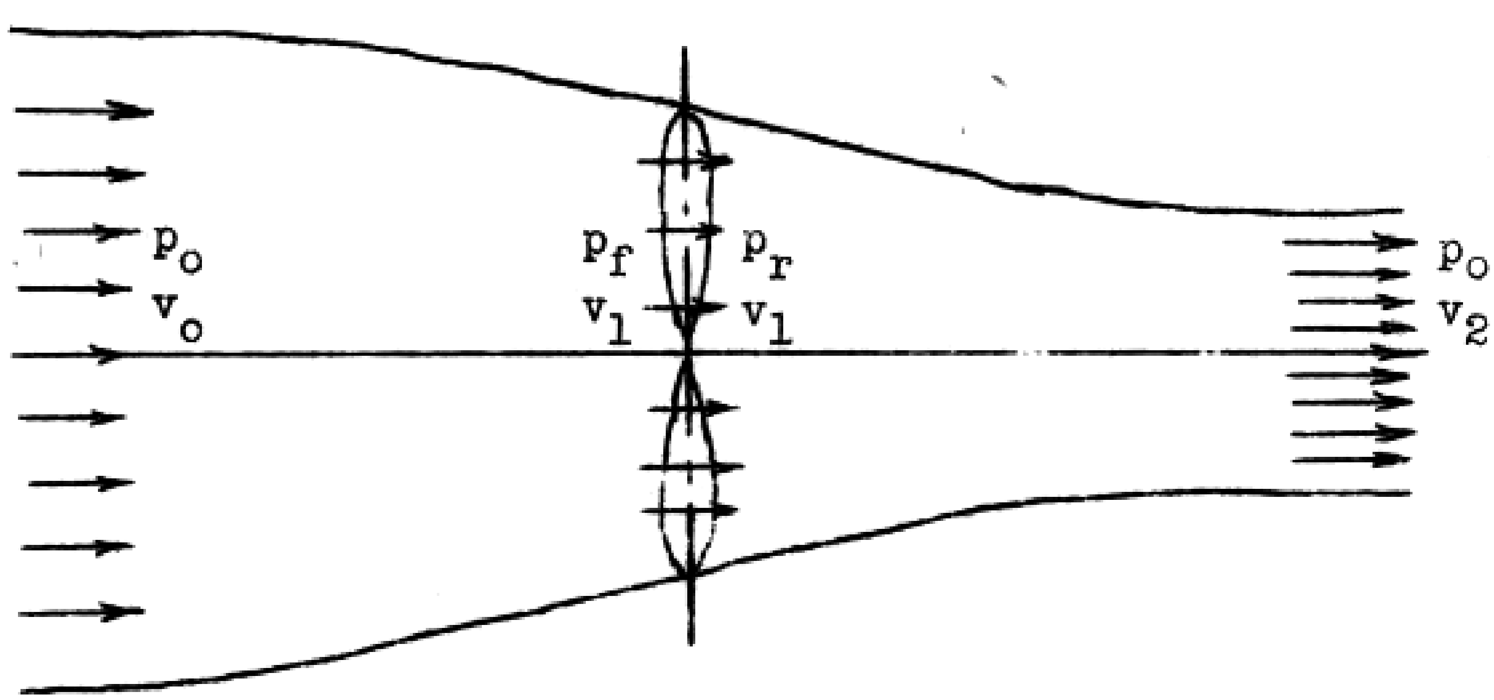


Figure 1.

The jet of Fig. 1 shall establish itself in a medium (the free atmosphere) which outside the jet is under constant pressure and either at rest or in uniform motion. The pressure differences in the jet are supposed to be small enough to consider the fluid as incompressible. Furthermore, while the jet is exactly in equilibrium at its boundary with the surrounding fluid, a pressure difference at the converging surface of the jet when appearing in the analysis may be allowed and may be qualitatively explained by the centrifugal forces of the curved streamlines providing a transition in a thin surface layer from the outside to the inside pressure.

For simplicity assume a uniform velocity distribution in each section; this assumption can be proved to give the highest efficiency.

The velocities v shall be the velocities relative to the propeller disc, so that in general v_o is equal to the forward speed of the propeller.

For the application of dynamic principles the absolute velocities of the jet left behind by the advancing propellers are needed. They are, for instance,

(1) $$v_{2,abs.} = v_2 - v_o .$$

The dynamic principles shall be applied now. The momentum increase per unit of time is equal to the force.

$$\frac{dmv}{dt} = F,$$

and in this special case $\frac{vdm}{dt} = F$. Therefore,

(2) $$\rho S v_1 (v_2 - v_o) = T$$

where S is the area of the propeller disc, T is the propeller thrust and $\rho S v_1$ is the increase of mass per unit of time in the slip stream

The engine power P driving the propeller is consumed partly by the work of the thrust T driving the propeller disc (or the ship) with velocity v_o and partly by the increase of kinetic energy of the slip stream per unit of time.

(3) $$\tfrac{1}{2} \rho S v_1 (v_2 - v_o)^2 + T v_o = P .$$

A third principle is needed to represent the assumption of the pressure jump in the plane of the propeller, which is equal to the thrust T divided by the area S. Here the Bernoulli energy equation of the two parts of a streamline in front and behind the disc shall be applied to give:

$$p_o + \tfrac{1}{2} \rho v_o^2 = p_f + \tfrac{1}{2} \rho v_1^2$$

$$p_o + \tfrac{1}{2} \rho v_2^2 = p_r + \tfrac{1}{2} \rho v_1^2 .$$

By subtraction we have

$$(4) \qquad p_r - p_f = \frac{T}{S} = \tfrac{1}{2}\rho(v_2^2 - v_o^2)$$

and using eq. (2)

$$2v_1(v_2 - v_o) = (v_2 - v_o)(v_2 + v_o)$$

$$v_1 = \tfrac{1}{2}(v_2 + v_o).$$

Or when expressed in absolute velocities, $v_1 = v_{1,abs.} + v_o$, and $v_2 = v_{2,abs.} + v_o$, we have

$$(5) \qquad v_{1,abs.} = \tfrac{1}{2}v_{2,abs.} \; .$$

That is, one half of the absolute velocity of the slip stream is already gained (by suction) at the plane of the propeller disc.

In this way the following two equations are sufficient to determine the characteristic action of the supposed ideal actuator.

$$(2a) \qquad T = \tfrac{1}{2}\rho S(v_2^2 - v_o^2)$$

$$P = \tfrac{1}{2}\rho S v_o(v_2^2 - v_o^2) + \tfrac{1}{2}(v_2 + v_o)(v_2 - v_o)^2$$

$$(3a) \qquad = \tfrac{1}{4}\rho S(v_2^2 - v_o^2)(v_2 + v_o).$$

From these results follows readily the efficiency

$$\eta = \frac{Tv_o}{P} = \frac{v_o}{\tfrac{1}{2}(v_2 + v_o)} = \frac{1}{\tfrac{1}{2}(1 + \frac{v_2}{v_o})} ,$$

or in absolute velocities

$$(6) \qquad \eta = \frac{1}{1 + \frac{\tfrac{1}{2}v_{2,abs.}}{v_o}} .$$

This expression shows that a high efficiency can only be obtained by a slip stream velocity small in comparison to the forward speed of the ship.

But equation (2a) shows that to obtain the required thrust a certain value of the product Sv_2^2 is necessary. Therefore, for high efficiency one has rather to choose a large area of propeller disc in order to avoid a high slip stream velocity. On the other hand, the blade theory of the propeller given in later chapters will restrict the area S by other considerations.

For very high travel velocity v_o higher slip stream velocity v_2 or $v_{2,abs.}$ are allowable according to equation (6) so that in such cases smaller propeller disc area S would be allowable (turbine propulsion).

Thrust Power Equation.

It is convenient to introduce into eqs. (2a) and (3a) the dimensionless variables

$$\tau = \frac{T}{\frac{1}{2}\rho S v_o^2}, \quad \text{a thrust variable,}$$

$$\varepsilon = \frac{P}{\frac{1}{4}\rho S v_o^3}, \quad \text{an engine power variable,}$$

and

$$\phi = \frac{v_2}{v_o}, \quad \text{a slip stream variable.}$$

Equations (2a) and (3a) will take the following forms:

$$\tau = \phi^2 - 1, \tag{2b}$$

$$\varepsilon = (\phi^2 - 1)(\phi + 1). \tag{3b}$$

Eliminating ϕ, ε can be expressed in terms of τ so that:

$$\varepsilon = \tau(1 + \sqrt{1 + \tau}). \tag{7}$$

Asking for thrust in terms of power an equation of the third degree would appear. Therefore it is more convenient to solve eq. (7) graphically by drawing the curve $\varepsilon = f(\tau)$. See Fig. 2.

This curve gives for any horsepower P, propeller disc area S, and airplane speed v_o the ideally obtainable highest thrust, but one must not forget that such ideal values are limited by considerations which will be treated later.

For a stationary propeller, as in the take-off of an airplane, in a helicopter, or for a fan, where v_o is zero, the

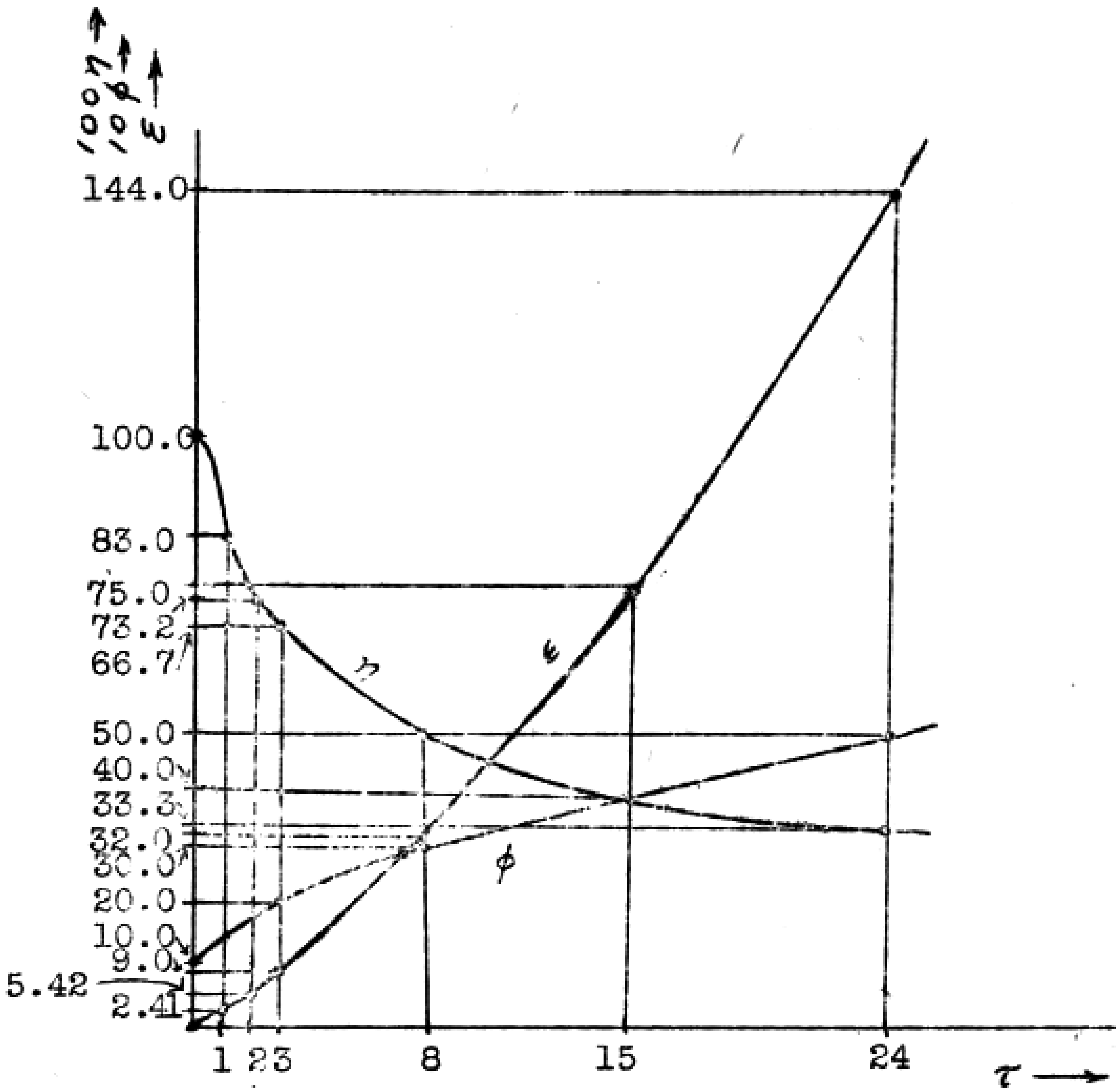


Figure 2. $\varepsilon = f(\tau)$.

dimensionless representation by ϕ, τ, and ε is not usable any more and the basic equations (2a) and (3a) must be used. Thus

$$T = \frac{1}{2} \rho S v_2^2 ,$$

$$P = \frac{1}{4} \rho S v_2^3 ,$$

so that the jet velocity v_2 and the thrust T are given by the power P_{avail}, the density ρ, and the disc area S in the form

(8) $$v_2 = \sqrt[3]{\frac{4P}{\rho S}}$$

and,

(9) $$T = \sqrt[3]{2P^2 \rho S} .$$

In the case of the airplane or helicopter the maximum of T is desired and one observes from eq. (9) that the thrust increases more slowly than the power, the density, and the disc area.

In the case of a fan the maximum of mass output per second is desired, that is

(10) $$\rho v_2 S = \sqrt[3]{4P\rho^2 S^2}$$

which shows that the accelerated mass increases more slowly with the power than the thrust, but that an increase in density and area is more effective than for the thrust.

Working against pressure the results for the fan become somewhat different if it does not work against free air $p_2 = p_o$ as assumed here, but into a room which is under higher pressure $p_2 > p_o$. This would change only the statements of the Bernoulli equation, which then become:

$$p_o + \tfrac{1}{2}\rho v_o^2 = p_f + \tfrac{1}{2}\rho v_1^2$$

$$p_2 + \tfrac{1}{2}v_2^2 = p_r + \tfrac{1}{2}v_1^2 .$$

$$p_r - p_f = \frac{T}{S} = \tfrac{1}{2}\rho(v_2^2 - v_o^2) + p_2 - p_o .$$

Equations (2) and (3) in this case must also be completed for the counterpressure p_2 in the slipstream or the pressure excess $\Delta p = p_2 - p_o$, viz.:

(2b) $$T = \rho S v_1 (v_2 - v_o) + \Delta p\, S$$

(3b) $$P = Tv_o + \tfrac{1}{2}\rho S v_1 (v_2 - v_o)^2 + \Delta p\, S v_1 .$$

The additional term in T gives the retarding force due to the counter pressure, while in P it takes into account the pressure energy of the jet after it has become cylindrical.

Comparing now the two expressions for $\frac{T}{\rho S}$ the following relation is obtained:

$$v_1(v_2 - v_o) + \frac{\Delta p}{\rho} = \tfrac{1}{2}(v_2^2 - v_o^2) + \frac{\Delta p}{\rho}$$

so that again as in (5)

(5b) $$v_1 = \tfrac{1}{2}(v_o + v_2).$$

The value v_1 from eq. (5b) must then be used to eliminate v_1 from equations (2b) and (3b). This procedure may be necessary for the case where it is required to furnish in a cabin for stratosphere flying a pressure excess.

The Windmotor.

The equations (2) and (3) are also applicable for the reversed flow through a windmill or a windmotor or a rotating windbrake or parachuting helicopter. Using the same dimensionless notations as for a propeller (only changing the sign of τ and ε) the performance is again given by

$$\tau = 1 - \phi^2$$

$$\varepsilon = (1 - \phi^2)(1 + \phi)$$

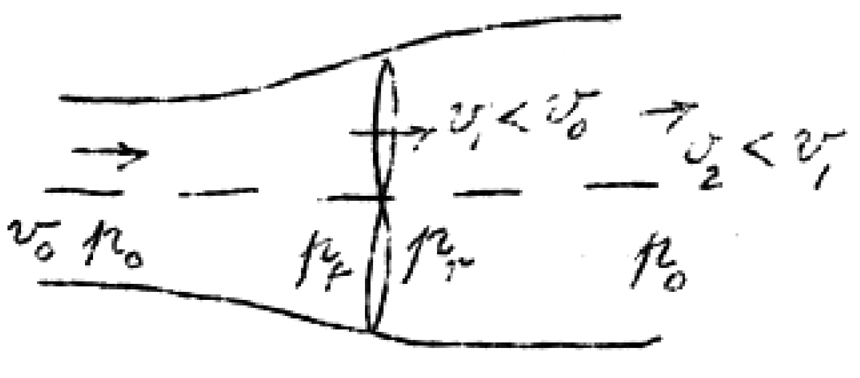


Figure 3.

Here the desirable maximum is the power gained for an appropriate value of the flow retardation ratio ϕ.

Therefore it may be demanded:

$$\frac{d\varepsilon}{d\phi} = 0 = -2\phi(1 + \phi) + 1 - \phi^2$$

$$= 1 - 2\phi - 3\phi^2$$

$$(\phi)_{\varepsilon_{max.}} = \frac{1}{3} \text{ or } -\frac{4}{3}.$$

$$\varepsilon_{max.} = \frac{32}{27},$$

$$(\tau)_{\varepsilon_{max.}} = \frac{8}{9}.$$

These results mean that the maximum power is gained if the wind velocity, or in a more general expression the relative inflow velocity, is retarded to one third of its value sufficiently far behind the wind motor and that then the power gained is given by

(11) $$\frac{P}{\frac{1}{4}\rho S v_o^3} = \frac{32}{27},$$

while the thrust produced, necessary to know for strength calculation is

$$\frac{T}{\frac{1}{2}\rho S v_o^2} = \frac{8}{9}.$$

The positive value of ϕ gives a maximum of ε, the second negative value a minimum of ε as one sees by the value of the second derivative. The negative value of ϕ that is a counter-flow in the region behind the motor would require some artificial realization but it is not desirable anyhow.

The general equations for a <u>non uniform distribution</u> in the slipstream, which is still assumed to be non-rotating, (engine torque balanced inside the propeller mechanism), may now be given. As will be shown, they prove that the highest thrust for a prescribed engine power is obtained for the uniform distribution.

For a non uniform velocity distribution the Bernoulli equation gives as before

$$v_1 = \tfrac{1}{2}(v_2 + v_o)$$

because for a streamline, just as in (4)

(4a) $$\frac{dT}{dS} = p_r - p_f = \tfrac{1}{2}\rho(v_2^2 - v_o^2) = \rho v_1(v_2 - v_o).$$

The general dynamic equations become therefore

(2c) $$T = \rho\int dS\, v_1(v_2 - v_o) = \frac{\rho}{2}\int dS(v_2^2 - v_o^2)$$

$$P = \frac{\rho}{2}\int dS\left[(v_2^2 - v_o^2)v_o + \tfrac{1}{2}(v_2 - v_o)^2(v_2 + v_o)\right]$$

(3c) $$= \frac{\rho}{4}\int dS(v_2^2 - v_o^2)(v_2 + v_o)$$

where v_2 is an arbitrary function of the coordinates, depending on the design of the mechanism.

The problem is now to determine v_2 as such a function of the coordinates that for a given power P the thrust T be a maximum. A well known method gives the following conditions:

$$\frac{\partial}{\partial v_2}\left[v_2^2 - v_o^2 - \frac{\lambda}{2}(v_2^2 - v_o^2)(v_2 + v_o)\right] = 0$$

$$\int dS(v_2^2 - v_o^2)(v_2 + v_o) = \frac{4P}{\rho} .$$

Hence $v_2 = v_2(\lambda, v_o)$ = const., and therefore the desired maximum is obtained by returning to the case of uniform velocity distribution, that is, to the case discussed in the previous chapter.

It may be mentioned here that if the approach velocity v_o is disturbed by the influence of the aircraft body so that v_o is a function of the coordinates then the jet velocity v_2 will also become a function of the coordinates, for from the first of the above conditions

$$v_2 = \frac{2 - \lambda v_o}{6\lambda} \pm \sqrt{\left(\frac{2 - \lambda v_o}{6\lambda}\right)^2 + \frac{v_o^2}{6}} .$$

If this expression for v_2 is inserted into the integral for the power, the Lagrangian coefficient λ is determined, and consequently v_2 as a function of v_o and of the limits of the power integral.

It may further be indicated that it is possible by means of a non uniform distribution of velocities and pressures in the cross-sections of the slipstream, produced by the design of the mechanism, to satisfy exactly the boundary condition of continuity of pressure at the boundary of the jet. This can be done in different ways. Naturally, distributions which give a thin transition layer and which approach a uniform distribution will be preferred.

Assume for instance that the mechanism is so constructed that at the boundary $p_r = p_f = p_o$. From (4a) at the boundary also $v_2 = v_o = v_1$.

From the relation preceding (5)

$$2v_1 = v_2 + v_o$$

and the two Bernoulli equations for p_r and p_f (See Fig. 1) it is readily seen that the requirement

$$p_r = p_f = p_o \text{ at the outer boundary}$$

has the consequence

$$v_2 = v_1 = v_o \text{ at the outer boundary.}$$

Therefore both conditions are covered by putting

(12) $$v_1 = v_o + v_{1,o}\sqrt{1 - \left(\frac{r_1}{R_1}\right)^2}\,.$$

This assumption is illustrated in Fig.(4).

Thrust and power in (2c) and (3c) in terms of v_1 instead of v_2 take the form:

(2d) $T = 4\rho\pi\int dr_1 r_1 v_1(v_1 - v_o)$

(3d) $P = 4\rho\pi\int dr_1 r_1 v_1^2(v_1 - v_o)$

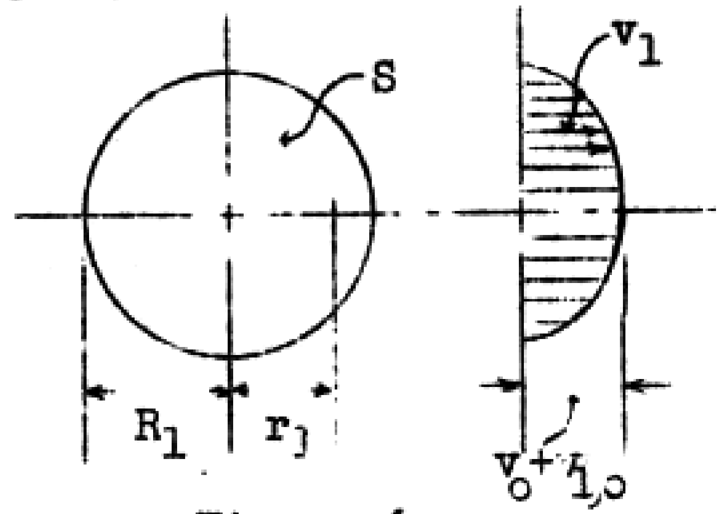


Figure 4.

and upon inserting the elliptical distribution given in (12), introducing $\frac{r_1}{R_1} = \sigma$, $\frac{v_{1,o}}{v_o} = \psi$ and again the thrust number τ and the power number ε. (2d) and (3d) become

(2e) $$\tau = 8\psi\int_0^1 \sigma d\sigma\sqrt{1-\sigma^2}\,(1 + \psi\sqrt{1-\sigma^2})$$

(3e) $$\varepsilon = 16\psi\int_0^1 \sigma d\sigma\sqrt{1-\sigma^2}\,(1 + \psi\sqrt{1-\sigma^2})^2$$

The integration, easily performed by putting $\sqrt{1-\sigma^2} =$ [illegible] leads to:

$$\tau = \frac{8}{3}\psi(1 + \frac{3}{4}\psi)$$

$$\varepsilon = \frac{16}{3}\psi(1 + \frac{3}{2}\psi + \frac{3}{5}\psi^2).$$

These equations can be discussed in the same way as (2b) and (3b), especially graphically as in Fig. (4).

It may be interesting to consider the relation between the elliptical velocity distribution on one hand and the corresponding thrust and power distributions on the other hand. The latter are given by the integrands of (2e) and (3e), viz.:

(13) $$\tau' \equiv \frac{d\tau}{d\sigma} = 8\psi\sigma\sqrt{1-\sigma^2}(1 + \psi\sqrt{1-\sigma^2})$$

(14) $$\varepsilon' \equiv \frac{d\varepsilon}{d\sigma} = 16\psi\sigma\sqrt{1-\sigma^2}(1 + \psi\sqrt{1-\sigma^2})^2.$$

These expressions, which depend somewhat on the choice of the velocity number $\psi = \frac{v_{1,o}}{v_o}$, that is, on the value of the total power $P = \varepsilon \times \frac{1}{4}\rho v_o^3 S$, are qualitatively shown in Fig. (5).

It is characteristic that the distribution curves approach the limit $\tau' = 0$, $\varepsilon' = 0$ for $\sigma = 1$ more suddenly than the velocity distribution and also with a tangent perpendicular to the radius R_1 ($\sigma = 1$).

These results of the assumption (12) are very similar to the results of the hydrodynamic blade theory treated in later chapters, and in fact the two can be made to coincide.

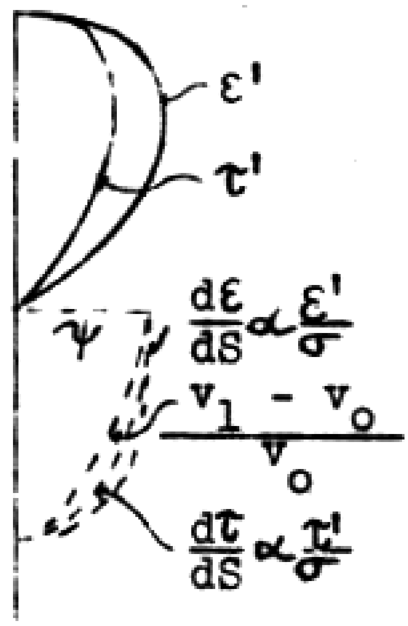


Figure 5.

Propelling Mechanism Producing Rotation of Slip Stream.

In the previous sections it was assumed that the torque in the propelling mechanism (actuator) was counterbalanced either by counterrotating propellers or by guide vanes either in front or behind the propeller.

More often the engine torque driving the propeller is not counterbalanced in this way, so that the moment of momentum law will demand a rotation of the slip stream, which, in certain cases indicated below, causes an appreciable loss of energy.

For the extension of the theory including slip stream rotation equations (2) and (3) must be completed and the moment of momentum equation must be added.

The basic equations become thus:

$$T = \rho S v_1 (v_2 - v_o), \tag{15}$$

$$P = T v_o + \tfrac{1}{2} \rho S v_1 \left[(v_2 - v_o)^2 + v_m^2\right], \tag{16}$$

where v_m is the average tangential velocity defined by the following expression

$$S v_m^2 = \int v_t^2 dS, \text{ where } v_t = v_t(r_2) \tag{17}$$

and the integral is taken over the v_2 section.

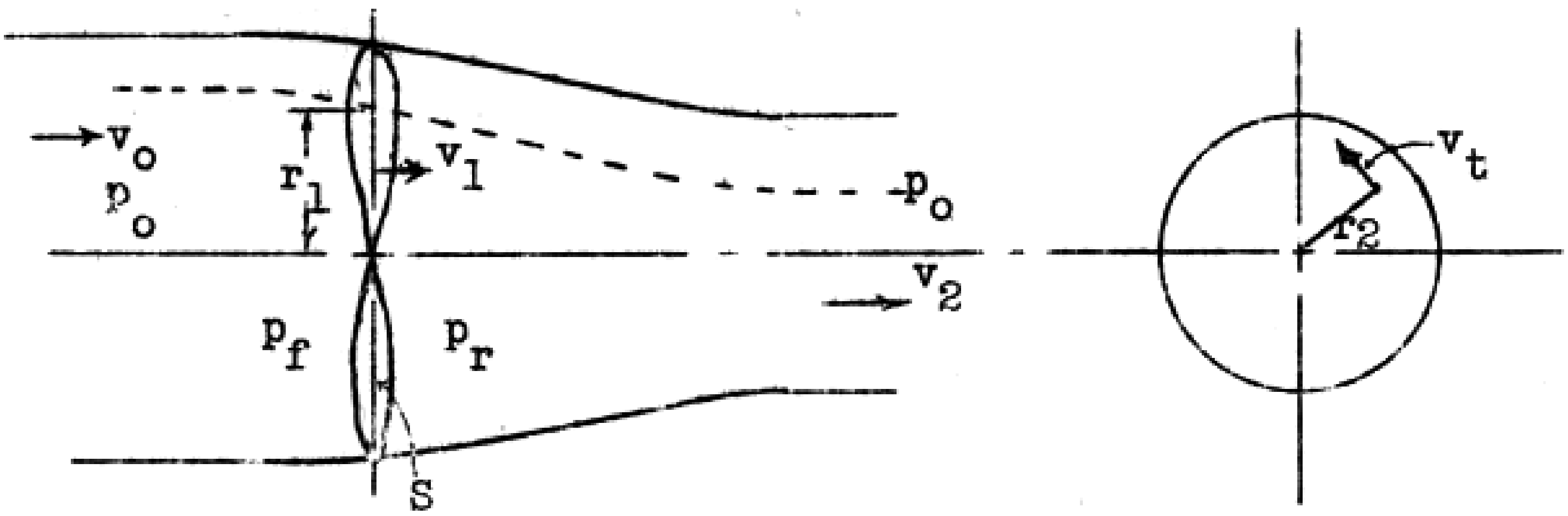


Figure 6.

If Q is the torque of the engine, the moment of momentum theorem gives

(18) $$Q = \rho S v_1 v_m r_m,$$

where r_m is the average radius of momentum defined by

(19) $$S v_m r_m = \int v_{t2} r_2 dS.$$

The theorem of moment of momentum gives also a relation between the circumferential velocities immediately behind the propeller disc and in the cylindrical part far behind if the friction between annular flow layers is neglected, namely,

(20) $$v_{t1} r_1 = v_{t2} r_2.$$

As the axial flow v_1 and v_2 is assumed to be uniform, there is a simple continuity relation between the radii and the axial velocities

(21) $$v_1 r_1^2 = v_2 r_2^2,$$

so that eqs. (14) and (16) can also be written

(22) $$S v_1 v_m^2 = v_2 \int v_{t1}^2 dS$$

(23) $$S v_m r_m = \int v_{t1} r_1 dS$$

where the integrals are now to be taken over the propeller disc area S_1.

A fourth equation is necessary, as in the preceding section without torque, to state the condition that the pressure discontinuity at the propeller disc plane produces the thrust.

The Bernoulli streamline gives

$$p_o + \tfrac{1}{2}\rho(v_2^2 + v_{t2}^2) = p_r + \tfrac{1}{2}\rho(v_1^2 + v_{t1}^2),$$

$$p_o + \tfrac{1}{2}\rho v_o^2 = p_f + \tfrac{1}{2}\rho v_1^2.$$

In the second equation no circumferential velocity appears because in front of the propeller there is no moment. Since from equations (17) and (18),

$$v_{t2}^2 = v_{t1}^2 r_1^2/r_2^2 = v_{t1}^2 v_2/v_1,$$

subtraction of the two equations yields the value of the thrust element:

$$dT = (p_r - p_f)dS = \tfrac{1}{2}\rho dS(-v_{t1}^2(1 - v_2/v_1) + v_2^2 - v_o^2)$$

Hence by integration, using (19)

$$T = \tfrac{1}{2}\rho S(v_m^2(1 - v_1/v_2) + v_2^2 - v_o^2).$$

Introducing again the dimensionless variables

$$\tau = T/\tfrac{1}{2}\rho S v_o^2,$$

$$\phi_1 = v_1/v_o,$$

$$\phi_2 = v_2/v_o,$$

and, $\phi_m = v_m/v_o$.

One obtains $\tau = \phi_m^2(1 - \phi_1/\phi_2) + \phi_2^2 - 1.$

Comparing with eq. (12) gives

$$\phi_m^2(1 - \phi_1/\phi_2) + \phi_2^2 - 1 = 2\phi_1(\phi_2 - 1)$$

or,

$$\phi_m^2 = \frac{2\phi_1(\phi_2 - 1) - (\phi_2 - 1)(\phi_2 + 1)}{1 - \phi_1/\phi_2}$$

$$= \frac{\phi_2(\phi_2 - 1)(2\phi_1 - \phi_2 - 1)}{\phi_2 - \phi_1}$$

(24) $$= \phi_2(\phi_2 - 1)\left(\frac{\phi_1 - 1}{\phi_2 - \phi_1} - 1\right),$$

which for non-rotation (m = 0) reduces to the former result, eq. (5).
Now observing that torque, power and the angular speed of the

propeller (ω) are related by

$$Q = P/\omega ,$$

and introducing the dimensionless quantity

(25) $$\phi_\omega = \omega r_m / v_o,$$

which can be described as an average circumferential relative speed of the propeller, one obtains from eq. (15) the final equation

(26) $$\varepsilon = 4\, \phi_1 \phi_m \phi_\omega .$$

The complete set of equations derived before may be repeated now.

(27a) $$\tau = 2\phi_1 (\phi_2 - 1),$$

(27b) $$\varepsilon = 2\tau + 2\phi_1 ((\phi_2 - 1)^2 + \phi_m^2),$$

(27c) $$\phi_m^2 = \phi_2 (\phi_2 - 1)\left(\frac{2\phi_1 - 1}{\phi_2 - \phi_1} - 1\right),$$

(27d) $$\phi_\omega = \varepsilon / 4 \phi_1 \phi_m .$$

Remark: In these equations the pressure decrease in the slipstream caused by the centrifugal force of the rotation of the slipstream expressed by

(28) $$p_2 = p_o - \rho \int_{r_2}^{R_2} \frac{dr_2}{r_2} v_{t,2}^2$$

has been neglected; this neglect implies that $v_{t,2} \ll v_2$, which can be verified by the numerical results of the equations (27a) to (27d) by means of the neglected additional terms, which for the thrust are

$$T_1 = -\int dS (p_2 - p_o)$$

(29) $$\tau_1 = -\frac{2}{S} \int dS \int_{r_1}^{R_1} \frac{dr_1}{r_1} \frac{v_{t1}^2}{v_o^2}$$

and for the power

$$P_2 = -\int dS v_1 (p_2 - p_o)$$

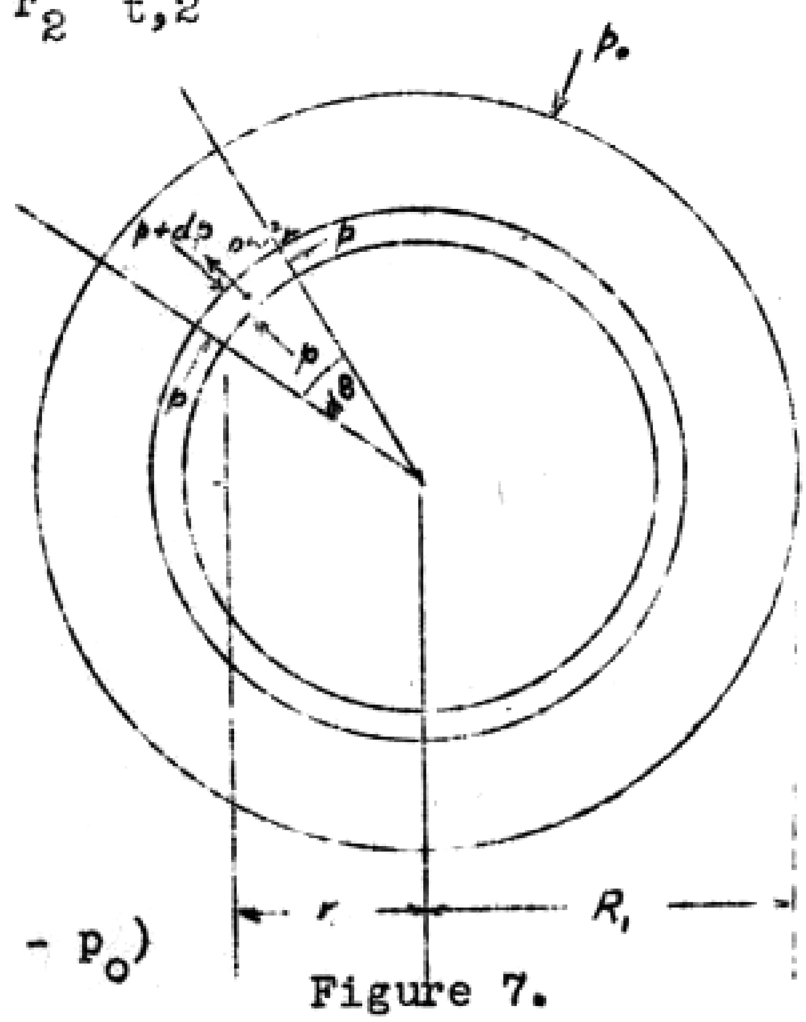


Figure 7.

$$\mathcal{E}_1 = -\frac{4}{S}\int dS\ \phi_1 \int_{r_1}^{R_1} \frac{dr_1}{r_1}\left(\frac{v_{t,2}}{v_o}\right)^2 \tag{30}$$

The effect of these neglected terms will partially cancel out in the efficiency expression because, as (29) and (30) show, they cause a decrease of thrust as well as of power. On the other hand, as both values decrease a greater disc area will be required by such a correction in order to absorb a prescribed engine power.

The problem presented by eq. 5 (27a) to (27d) may now be discussed.

In general the density ρ (altitude) the forward speed v_o, the disc area S, the engine power P and the angular velocity ω (number of revolutions per second $n = \frac{\omega}{2\pi}$) will be prescribed and those values of v_1, v_2, v_m and r_m which give a maximum thrust T will be sought.

In the dimensionless variable defined here this means that the values of the power variable $\mathcal{E}$ and the advance

$$J = \frac{v_o}{\omega R_1}$$

are prescribed and such values of the velocity variables ϕ_1, ϕ_2, ϕ_m and the relative mean radius $\frac{r_m}{R_1} = r'_m$ are desired which give a maximum of the thrust variable τ.

Eliminating τ and ϕ_m from (27b) by means of (27a) and (27c) one has the expression for the power variable ε:

$$\mathcal{E} = \frac{2\phi_1^2(\phi_2 - 1)^2}{\phi_2 - \phi_1}, \tag{27e}$$

from which ϕ_1 can be expressed by ϕ_2 and $\mathcal{E}$ in the form

$$\phi_1 = \frac{-\varepsilon \pm \sqrt{\varepsilon^2 + 8\varepsilon\phi_2(\phi_2 - 1)^2}}{4(\phi_2 - 1)^2} \tag{31}$$

and inserted in (27a) so that:

$$\tau = \frac{-\varepsilon \pm \sqrt{\varepsilon^2 + 8\,\varepsilon\,\phi_2(\phi_2 - 1)^2}}{2(\phi_2 - 1)} \tag{32}$$

As ε is prescribed and ϕ_2 is now the only remaining variable which can be chosen the maximum of τ must be found by putting

$$\frac{d\tau}{d\phi_2} = 0 \,.$$

This leads to $$\varepsilon^2(\phi_2 - 1)^2\left[2(\phi_2 - 1)^4 - \varepsilon(2\phi_2 - 1)\right] = 0$$

A zero value of the factor $\phi_2 - 1$ does not give a useful result as it leads to the triviality $\varepsilon = 0$ if inserted in (27b). The condition that the factor in brackets is zero may be written:

$$\varepsilon = \frac{2(\phi_2 - 1)^4}{2\phi_2 - 1}\,, \quad \varepsilon' = \frac{4(\phi_2 - 1)^3(3\phi_2 - 1)}{(2\phi_2 - 1)^2} \tag{33}$$

and it is represented graphically in Fig. 8

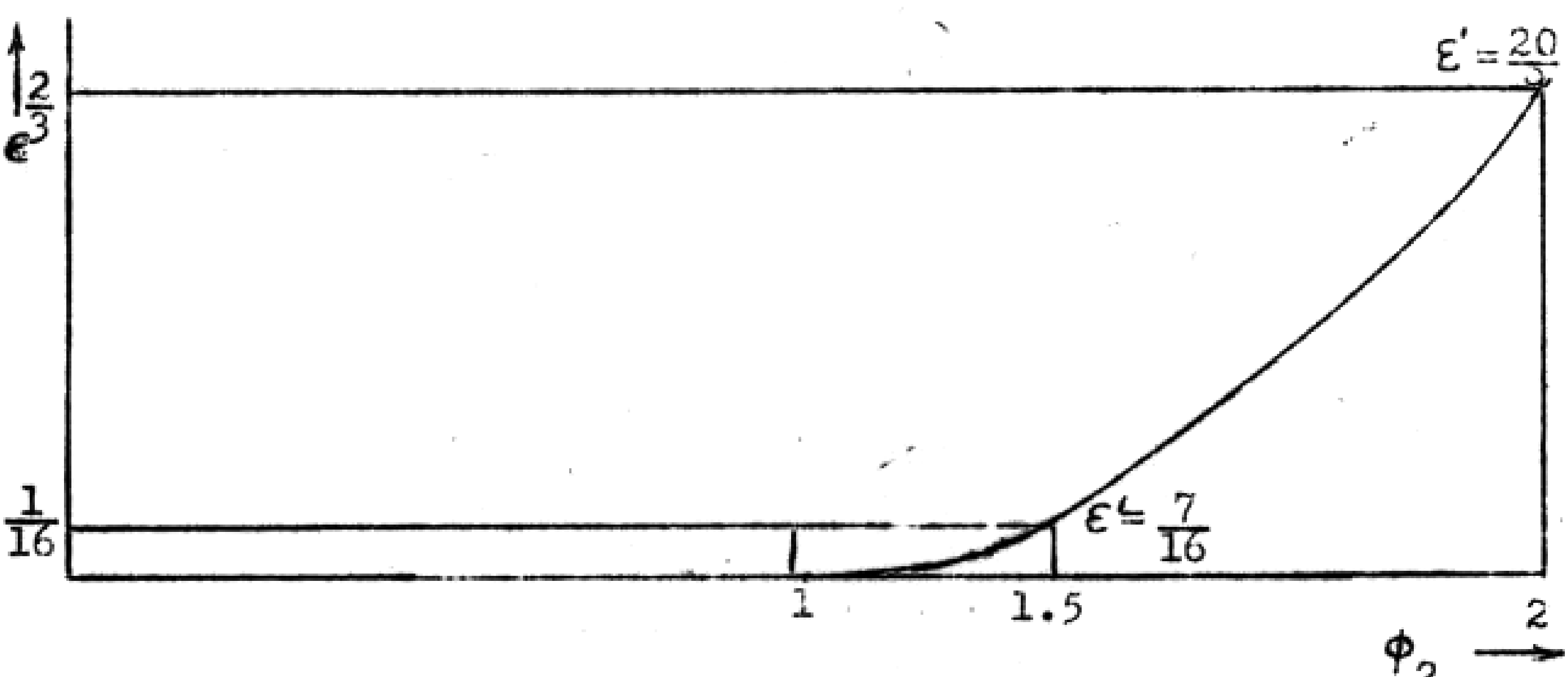


Figure 8.

ϕ_2 being given as a function of ε by (33) for the propeller of maximum thrust, (31) gives the corresponding values of ϕ_1 and (27c) the corresponding values of ϕ_m (the average tangential velocity). Finally the resultant relative radius $\sigma_m = \frac{r_m}{R_1}$

ascribed to ϕ_m is given by (27d) in the form:

(27d) $$\phi_\omega J = \sigma_m = \frac{J\phi}{4\,\phi_1\phi_m}$$

in which the value of the advance characteristic $J = \frac{v_o}{\omega R_1}$ appears, which was assumed to be prescribed by the tip radius R_1, the forward speed v_o and the rev.p.m. $J = \frac{60\omega}{2\pi}$.

Therefore it is possible to construct sets of curves giving the thrust variable τ, the efficiency $\eta = \frac{2\tau}{\varepsilon}$, the axial jet velocity variables ϕ_2 and ϕ_1, the tangential velocity variable ϕ_m and the radius variable σ_m as functions of the advance variable $N = \frac{v_o}{\omega R_1}$ with the power variable ε as parameter.

Of technically interesting questions which can be answered by these sets of curves the following may be mentioned:

What are the efficiencies η of

(1) a slipstream rotating due to the advance value $J = \frac{v_o}{\omega R_1}$

and (2) a slipstream non rotating due to counterbalancing of the propeller, where the power variable ε is equal in both cases?

Near which values of the advance and power variables lie the greatest efficiency losses due to rotation?

If equal efficiency is required for both cases how great is the percentage of saving in disc area S or disc radius R_1 of the counterbalanced propeller compared to the single propeller?

If the average tangential velocity variable $\phi_m = \frac{v_m}{v_o}$ and the corresponding radius variable $\sigma = \frac{r_m}{R_1}$ are found which tangential velocity distributions produce these two average values?

THE BLADE THEORY OF THE SCREW PROPELLER

In the preceding chapters the essential dynamic features of propeller action have been treated without making any suppositions about the special propelling mechanism. Results on the velocities necessary to obtain maximum values of thrust or power or mass output were obtained but the structural means of enforcing these velocities could not be given.

In the following chapters the theory of the Screw Propeller as the mechanism nearly exclusively applied for propulsion of aircraft and other ships shall be outlined.

The Screw Propeller is a system of blades or wings forced to rotate about an axis substantially perpendicular to their centroid (longitudinal) axes, this system at the same time advancing substantially in the direction of the driving axis (the drive shaft).

The theory of such a system must be a generalization of the wing theory of the airplane.

The principles used will be, just as in the wing theory, the conservation of circulation and vortex strength, the notion of bound and free vortex sheets, the Biot-Savart integral for the induction of velocities by vortex distribution and the expressions of lift and drag coefficients.

The exact hydrodynamic problem of a wing system in helical motion is mathematically much more complicated than in straight translation, even if, as at present, the theory confines itself to a first approximation of induced velocities very small relative to the undisturbed relative velocity at infinity and of the concentration of the wing to a bound vortex line. Nevertheless, the theory with the addition of some other approximative considerations has given already useful indications for technical developments.

Fig.(10a) shows the front view of a propeller system and a cross section of a blade on a circle of radial distance r, while the outside or tip radii are given as r_e.

The fluid is flowing against this cross-section with a relative resultant velocity V and the resultant force exerted by the fluid on this blade element cdr (see Figure (9)) will be perpendicular to the direction of the resultant inflow velocity V.

This direction is obtained by adding to the undisturbed inflow angle α the induced change of inflow angle i and, in order to obtain the effect of the parasite drag, a drag angle δ.

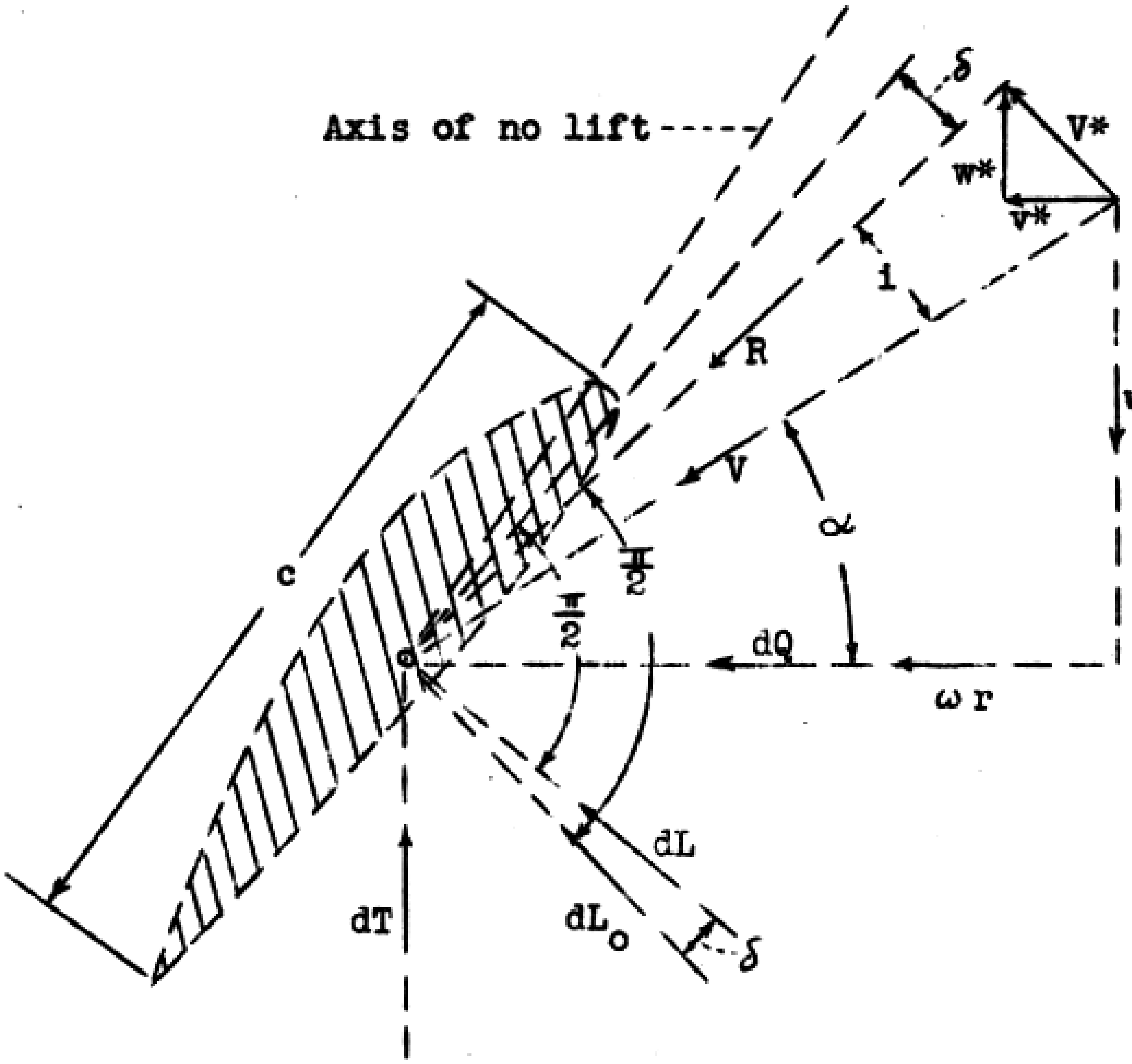


Figure 9.

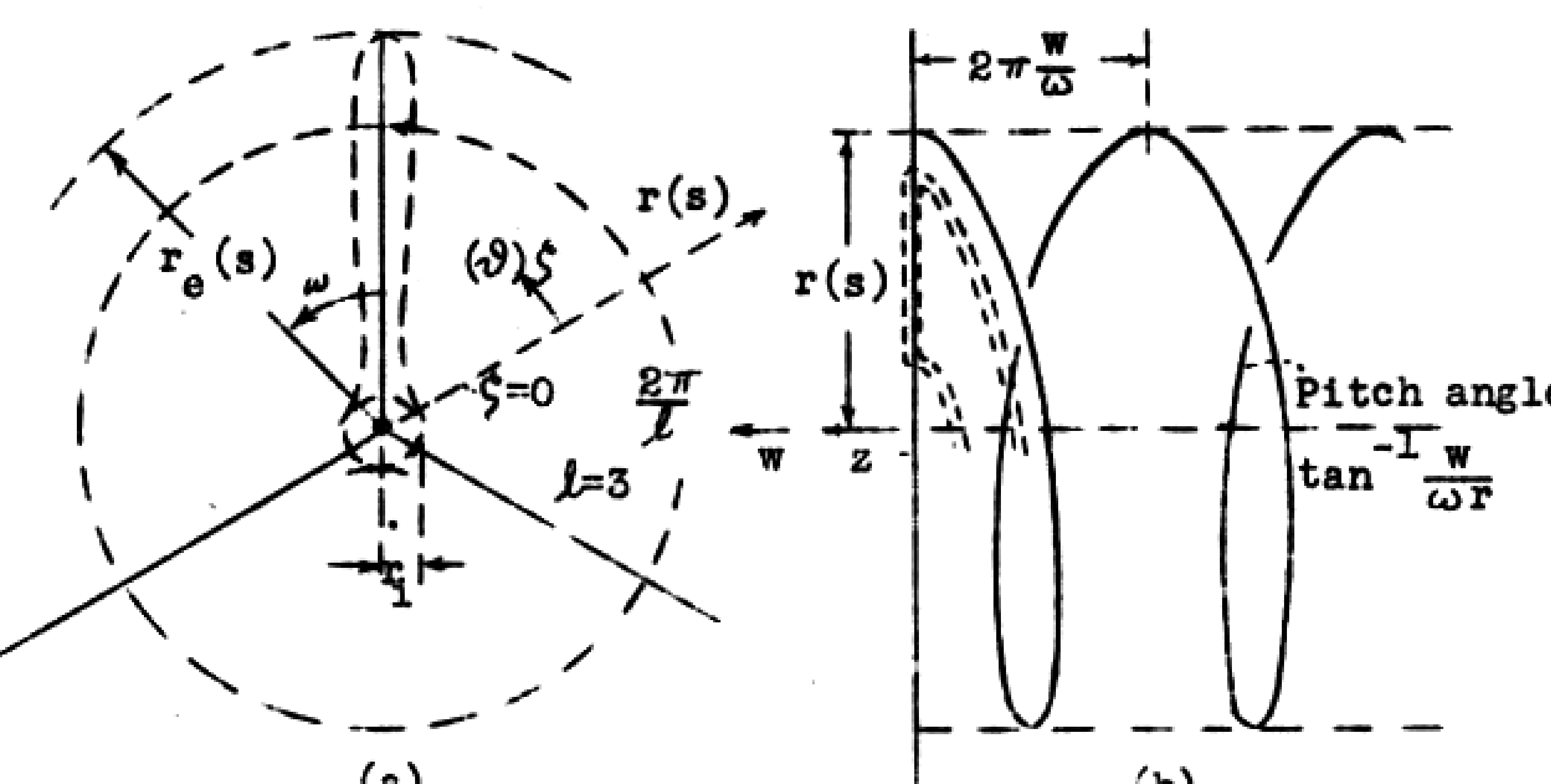


Figure 10.

The resultant force element dL (lift element) can then be written in the form:

(34) $$dL = C_L q c dr$$

where $q = \frac{\rho}{2} V^2$ is the dynamic pressure,

and according to experimental results about the retardation of the velocity by a parasite (form) drag angle δ it may be assumed

(35) $$V^2 = V_o^2 (1 - k\delta)$$

where (see Fig.(9))

(36) $$V_o^2 = w^2 + \omega^2 r^2$$

and k is an experimental coefficient of the approximate magnitude $k \approx 7.5$.

The reason for putting the velocity V_o (without friction drag) equal to the undisturbed velocity follows from dynamic considerations which show that the additional induced velocity is perpendicular to the direction of the disturbed inflow; this result for the small induced angle i permits neglect of the velocity change caused by induction.

From (34) the element dT of the thrust component and the element dQ of the torque component follow in the form:

(37) $$dT = dL \cos(\alpha + i + \delta)$$

(38) $$dQ = dLr \sin(\alpha + i + \delta).$$

From these relations follows as local energy loss

(39) $$dE = dQ\omega - dTw = dLw \frac{\sin(i + \delta)}{\sin\alpha}$$

and the "local" efficiency

(40) $$\eta_{loc.} = \frac{dTw}{dQ\omega} = \frac{\tan\alpha}{\tan(\alpha + i + \delta)} .$$

This result is quite analogous to the efficiency of a screw thread of pitch angle α and of angle of friction $f = i + \delta$ applied to lift a weight dT by means of a torque dQ, but while the angle of friction of such a hoist mechanism is a property of material and lubrication the analogous angle here is a complicated function of the vortex distribution in the whole fluid region.

From the expressions (37) and (38) one may now proceed to the values of the total thrust T and the total torque Q by integration, as the mutual influence of all elements is taken into account by the induced angle values i not yet known but to be determined.

This means, by introducing $i + \delta$, each blade element can be treated as an element in a two dimensional flow (of a wing of infinite aspect ratio).

Therefore

$$T = l\int_{r_i}^{r_e} C_L \frac{\rho}{2}V_o^2(1 - k\delta)c\,dr \cos(\alpha + i + \delta)$$

$$Q = l\int_{r_i}^{r_e} C_L \frac{\rho}{2}V_o^2(1 - k\delta)cr\,dr \sin(\alpha + i + \delta),$$

where l is the number of blades.

These relations may be transformed somewhat by introducing a dimensionless radial abscissa

$$s = \cot\alpha = \frac{\omega r}{w}$$

so that $$V_o^2 = w^2(1 + (\frac{\omega r}{w})^2) = \frac{w^2}{\sin^2\alpha}, \text{and}$$

further a dimensionless relative blade width

$$b = \frac{lc}{2\pi r}.$$

The angle $i + \delta$ is always so small

$$\cos(i + \delta) \sim 1 \quad \text{and} \quad \sin(i + \delta) \sim i + \delta.$$

Relations (37) and (38) become now:

(41) $$T = \pi\rho\frac{w^4}{\omega^2}\int_{s_i}^{s_e} bC_L ds(s - (i + \delta))\sqrt{1 + s^2}(1 - k\delta)$$

(42) $$Q = \pi\rho\frac{w^5}{\omega^3}\int_{s_i}^{s_e} bC_L ds\, s(1 + s(i + \delta))\sqrt{1 + s^2}(1 - k\delta)$$

bC_L may be called the relative lifting capacity of the blade system per unit of blade length.

The quantity BC_L and the outside and inside radii of the blades can be considered as the structural design functions of the propeller system.

Another representation better applicable to the hydrodynamic theory expresses the lift coefficient C_L in a well known manner by the circulation around the blade cross section, leaving out the parasite drag, by means of the Kutta-Joukowski theorem, viz:

From: $$dL_o = C_{L,o} \frac{\rho}{2} V_o^2 \, c\,dr = \rho \Gamma V_o dr$$

follows:

(43) $$C_{L,o} \, cV_o = 2\Gamma.$$

One may introduce a dimensionless circulation variable

(44) $$\Gamma^* = \Gamma \frac{\omega l}{w^2}$$

and also dimensionless force coefficients as follows:

a thrust coefficient

(45) $$C_T = \frac{T\omega^2}{\rho w^4}$$

torque and power coefficients

(46) $$C_Q = \frac{Q\omega^3}{\rho w^5} = C_P = \frac{P\omega^2}{\rho w^5} \text{ respectively}$$

an energy loss coefficient

(47) $$C_E = \frac{E\omega^2}{\mu w^5} .$$

Then

(45a) $$C_T = \int ds \, \Gamma^* \left[s - (i + \delta) \right] (1 - k\delta)$$

(46a) $$C_Q = \int ds \, s \, \Gamma^* \left[1 + s(i + \delta) \right] (1 - k\delta)$$

(47a) $$C_E = \int ds \, \Gamma^* \, (i + \delta)(1 + s^2)(1 - k\delta) .$$

The essential problem of a hydrodynamic blade theory of the propeller consists in the determination of the distribution of circulation Γ around the blades corresponding to a distribution of the induced inflow angle change i at and along the blade axes and vice versa.

If one contents oneself with a knowledge of the mean inflow angle change i_m on each circumference instead of the local angle change i at each blade section the solution is not difficult. This average solution gives a good approximation for a propeller system of many narrowly spaced blades and has even been approximately adapted to small blade numbers .

This simplification may be given as a preliminary introduction to the physical meaning of the problem, though it makes each value of i_m depend only on the value of Γ at the same cross-section and it shows only by an additional somewhat artificial adaptation the necessary approach of Γ towards zero at the tip.

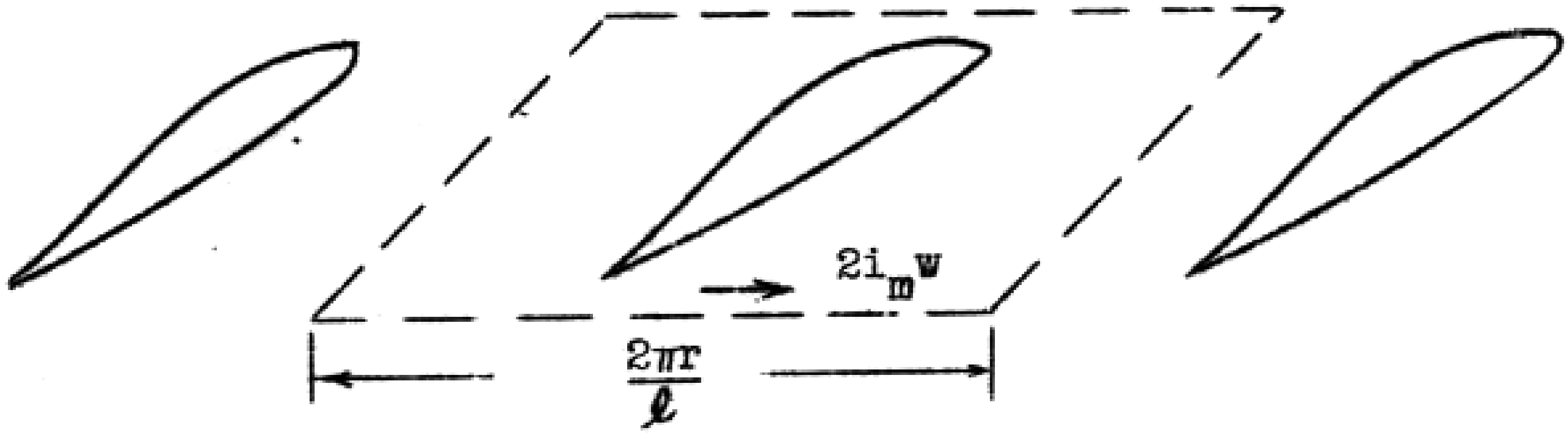


Figure 11.

Fig. (11) shows the arrangement of the blade sections on the circle of radius r developed in a plane with a circuit around one blade section on which the circulation Γ will be calculated.

Obviously only the part of the circuit lying on the plane of rotation behind the trailing edge of the blade gives a contribution to the circulation Γ, because in front of the blade there is no average circumferential velocity and the contribution on the curves along the flow between the blades cancel each other on account of symmetry.

Now the circumferential velocity behind the blade according to Fig. (11) is $v^* = 2V^* \frac{w}{V} = 2i_m w$.

The factor 2 must be introduced because the induced tangential velocity v* increases from zero in front of the blade to the value 2v* behind the blade and because the mean value in the axial direction determines the lift force dL.

Therefore:

$$\Gamma = 2i_m w 2\pi r/\ell$$

$$\Gamma^* = \Gamma \frac{\omega \ell}{w^2} = 2i_m \frac{\omega}{w} 2\pi r = 4\pi i_m s.$$

Equations (45), (46) and (47) can now be written in the form:

(45b) $$C_T = 4\pi \int_{s_i}^{s_e} ds\, s i_m \left[s-(i+\delta) \right] (1-k\delta),$$

(46b) $$C_Q = 4\pi\int_{s_i}^{s_e} ds\ s^2 i_m \left[1 + s(i + \delta)\right](1 - k\delta)$$

(47b) $$C_E = 4\pi\int_{s_i}^{s_e} ds\ s\ (1 + s^2) i_m (i + \delta)(1 - k\delta)$$

One can conclude from general considerations that the difference between i_m and i must decrease with the number of blades, that is must decrease towards the root of the blade, and that i_m must become zero for $s = s_e$, that is, at the tip of each blade, because the circulation, just as in the wing theory, must be zero on account of the Helmholtz theorem that the circulation or vortex strength can not end abruptly.

In order to make the mean angle change i_m more nearly equal to the local angle change a nozzle ring enclosing the propeller so as to hinder the radial escape of the flow from the pressure side around the tip to the suction side of the blade (see Figure 12) has been suggested and recently tried for marine propellers with some success.

To apply such a device for air propellers seems to lead to cumbersome structural details.

The last set of equations varies the problem of a hydrodynamic treatment insofar that instead of the relation between the distributions of local angle i and circulation Γ, the relation between i and its mean value distribution is sought. This change of question may give an additional physical aspect but mathematically it is only another method of expression.

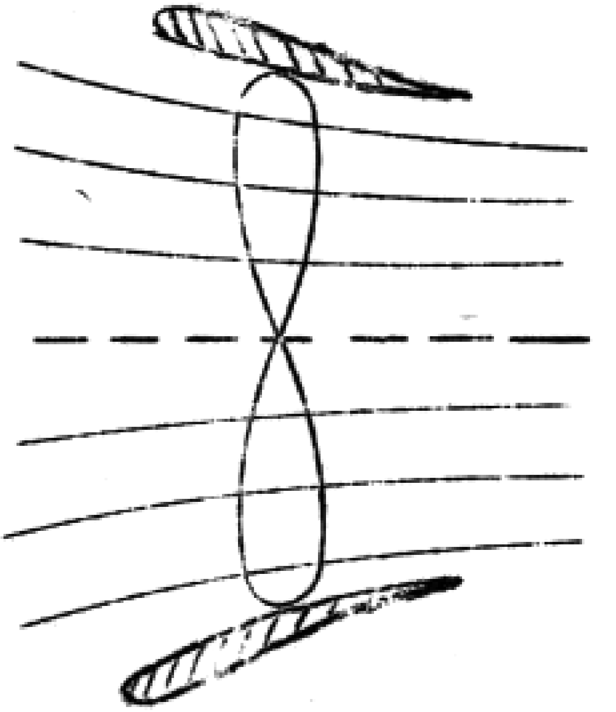


Figure 12.

As in the theory of the wing, the relation between circulation Γ or, more exactly, between rate of change of circulation $\frac{d\Gamma}{dr}$ and induced angle i must be found by considering a vortex sheet of strength $\frac{d\Gamma}{dr}$ left behind the trailing edge of each blade coinciding with the streamlines.

The Theorem of A.Betz of Minimum Induced Energy Loss.

Analogously to the vortex sheet theory of the wing of finite aspect ratio a rotating blade system of finite radial span producing a lift force and its components of thrust and torque (see equations (41) and (42)) leaves behind its trailing edges free vortex sheets of strength $\frac{d\Gamma}{dr}$ carried along with the streamlines and inducing additional velocities, the components of which perpendicular to the undisturbed inflow charge the angle of inflow as expressed in the equations just quoted.

For a first approximation of the mathematically complicated proposition it will be assumed that the vortex sheet is following the streamlines relative to the blade system as they would appear in a flow not changed by the induced velocities, which presupposes that the induced velocities V* in Figure (9) are very small relative to the undisturbed velocities V. One has with this approximation a set of helical vortex sheets of constant pitch and edge radius.

As a second approximation it has been suggested and in some cases calculated that the trailing helical vortex sheets, while having still a constant pitch and edge radius, be considered lying on the streamlines immediately behind the trailing edges of the blades, as these streamlines are changed by the induced velocities.

This second approximation would take into account only the beginning of the deformation of the streamlines but would neglect the deformation proceeding in the wake, relying on the fact that only the induction of vortices relatively near to the propeller disc can make an appreciable contribution.

The first approximation covers only the case of values $s_e = \frac{\omega r_e}{w}$ sufficiently far from the stationary or take off running of propeller, this second approximation permits a better approach to the conditions where the rotational speed is large relative to the forward speed.

The theorem of Betz is based on the described first approximation assuming that i and $i_m = \frac{\Gamma^*}{4\pi s}$ are small relative to unity and that the vortex lines left behind the trailing edges are helical curves of constant pitch lying on circular cylindrical surfaces. It may be stated as follows:

A screw propeller of any number of blades symmetrically arranged gives the minimum induced energy loss for prescribed torque or thrust if the distribution of circulation along the blades is such that the induced velocities are the same as would be produced by a rigid helical sheet moving with an axial or rotational velocity.

The minimum proposition may be stated by referring to equations (47a) and (46a) as follows:

$$C_E = \text{minimum}$$

under the condition that

$$C_Q = C_{Q,\text{prescribed}}.$$

The theorem itself may be expressed by the following formulae.

The induced angle distribution $i = i(s)$ which gives the minimum induced energy loss is given by

(48) $$(i)_{C_{E,\min.}} = \frac{w_1}{w}\frac{s}{1+s^2}, \quad \frac{w_1}{w} = \lambda << 1$$

where w_1 the axial velocity of the rigid helical sheets and w the axial component of the undisturbed inflow velocity.

The minimum energy loss coefficient takes then the value

(49) $$C_{E,\min.} = \lambda \int_{s_i}^{s_e} ds\, \Gamma^* s \;.$$

That (48) actually represents the condition of the rigid axially moving sheet, namely that the velocities at its surface perpendicular to it are equal to the velocities of the surface itself perpendicular to it can be seen from Figure (13).

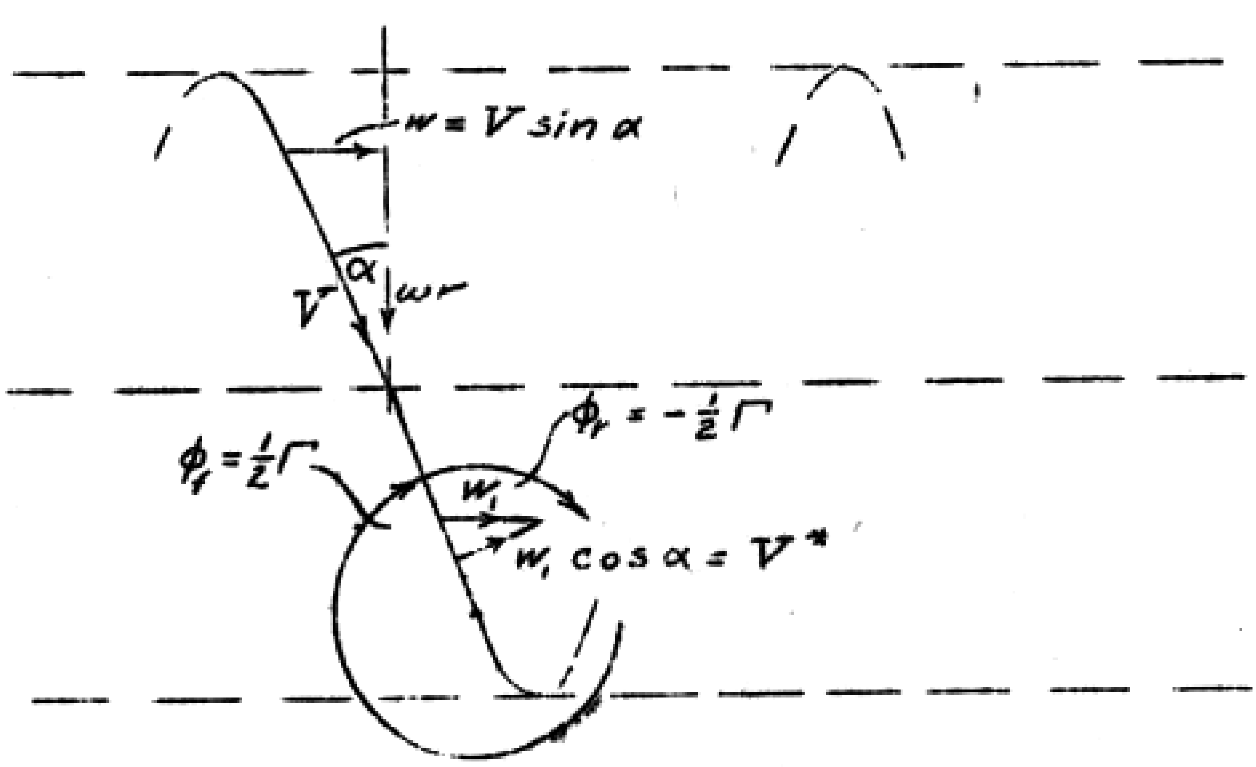


Figure 13.

The proof can be given in a form much simpler and more convincing than the original proof of Betz. It follows the method given by R.Fuchs for the corresponding theorem of Munk for the straight moving wing.

We start from a reciprocal theorem of Gauss stating that for two potential functions ϕ_1 and ϕ_2 the following two integrals both taken over the same closed surfaces are equal viz.

$$\int dS\, \phi_1 \frac{\partial \phi_2}{\partial n} = \int dS\, \phi_2 \frac{\partial \phi_1}{\partial n}$$

where n is the direction of the normal defined positive towards the interior of the closed surface.

This surface may be given by two cross sections far behind the propeller disc, by two circular cylindrical surfaces the one at the radial number s_1, the other at a sufficiently great radial distance, and by two adjacent helical surfaces between the two cross sections.

To the integrals over this closed surface only the helical parts give a contribution. In fact since ϕ and the velocities $\frac{\partial \phi}{\partial z}$ have become independent of the distance z of the cross sections, since $\frac{\partial \phi}{\partial r}$ on the inner cylindrical surface is zero and since $\phi \frac{\partial \phi}{\partial r}$ on the outer cylindrical surface decreases more rapidly than dS increases the integrals need to be taken only over the helical surfaces on which the velocities $\frac{\partial \phi}{\partial n} = V^*$ and the potentials ϕ are both of opposite signs.

For the potentials this is the case, because the circulation over the edge of a helical sheet enclosing the vortex strength down to the point considered, must be equal to Γ and at this surface of discontinuity the potential ϕ must jump from $-\frac{\Gamma}{2}$ to $+\frac{\Gamma}{2}$.

Therefore the Gauss theorem for this particular case can be written in the form

$$(50) \qquad \int_{s_1}^{s_e} \Gamma_1^* \, i_2(1 + s^2)ds = \int_{s_1}^{s_e} \Gamma_2^* \, i_1(1 + s^2)ds$$

where $\frac{\partial \phi}{\partial n} = V^*$, $dS = dzdr \frac{V}{w} = dzdr(\sin\alpha)^{-1}$

$$s = \frac{\omega r}{w} = \cot\alpha\,, \quad V^* = iV = iw(\sin\alpha)^{-1}, (\sin\alpha)^{-2} = 1 + s^2$$

Equations (50) referring to (47a) may also be written

$$C_{E,1,2} = C_{E,2,1}$$

We consider now the energy loss due to the difference of two arbitrary circulations Γ_1, Γ_2 and the difference of the two corresponding induced angles i_1, i_2

$$C_{E,1-2} = \int ds(\Gamma_1{}^* - \Gamma_2{}^*)(i_1 - i_2)(1 + s^2)$$

$$= C_{E,1} + C_{E,2} - \int ds(\Gamma_1{}^* i_2 + \Gamma_2{}^* i_1)(1 + s^2)$$

$$= C_{E,1} + C_{E,2} - 2\int ds\,\Gamma_2{}^* i_1(1 + s^2)$$

We choose then for i_1 the distribution (48) induced by a rigid helical sheet so that

$$C_{E,1-2} = C_{E,1} + C_{E,2} - 2\lambda\int ds\; s\,\Gamma_2{}^*$$

Now the condition of prescribed torque must be introduced

$$C_Q = \int ds\,\Gamma_2{}^* s(1 + si_2) = \int ds\,\Gamma_1{}^* s(1 + si_1)$$

$$= \int ds\,\Gamma_1{}^* s\left(1 + \lambda\frac{s^2}{1 + s^2}\right)$$

This leads to

$$2\lambda\int ds\,\Gamma_2{}^* s = 2\lambda\int ds\,\Gamma_1{}^* s + 2\lambda^2\int ds\,\Gamma_1{}^*\frac{s^3}{1 + s^2} - 2\lambda\int ds\,\Gamma_2{}^* s i_2\,.$$

But the premise was that $\lambda = \frac{w_1}{w}$ and i should be small quantities and therefore the two last integrals on the right side must be neglected, so that the relation appears in the form:

$$2\lambda\int ds\,\Gamma_2{}^* s = 2C_{E,1}$$

and therefore the energy loss due to the differences becomes

$$C_{E,1-2} = C_{E,1} + C_{E,2} - 2C_{E,1}$$

or

$$C_{E,1} = C_{E,2} - C_{E,1-2}\,.$$

This last result proves the theorem asserted, because $C_{E,1-2}$ being a positive quantity the particular energy loss corresponding to the rigid helical sheet is shown to be smaller than any other possible energy loss.

The theorem is remarkable because it gives the induced angle distribution i(s) but not the vortex distribution $\frac{d\Gamma^*}{ds} = \Gamma^{*\prime}(s)$ producing this angle distribution, so that the real mathematical difficulty consists in the determination of this particular $\Gamma^{*\prime}(s)$.

Before proceeding further it may be shown that the theorem of the solidified vortex sheet may be extended to include the profile drag angle.

For this extension it may be remembered that the hydrodynamic airfoil theory considers the fluid flow as a non potential flow only in the boundary layer of the airfoil so that for the vortex sheets behind the propeller disc the reciprocal theorem is still valid but with velocity square V^2 decreased to $V^2(1 - k\delta)$.

The energy loss due to a difference of two arbitrary circulations and corresponding angles i is given by

$$C_{E,1-2} = C_{E,1} + C_{E,2} - \int ds(\Gamma_2^*(i_1 + \delta) + \Gamma_1^*(i_2 + \delta))(1+s^2)(1-k\delta)$$

which by use of the reciprocal theorems if one neglects products of higher order than $\Gamma^* i$ becomes

$$C_{E,1-2} = C_{E,1} + C_{E,2} - 2\int ds\, \Gamma_2^*(i_1 + \frac{\delta}{2})(1 + s^2)(1 - k\delta)$$

$$- \int ds\, \Gamma_1^* \delta(1 + s^2)(1 - k\delta)$$

Assuming analogously to the assumption made above

$$(1 - k\delta)(i_1 + \frac{\delta}{2}) = \lambda \frac{s}{1 + s^2}$$

we get

$$C_{E,1-2} = C_{E,1} + C_{E,2} - 2\lambda\int ds\, \Gamma_2^* s - \int ds\, \Gamma_1^* \delta(1 + s^2)$$

in which finally the condition of prescribed torque coefficient C_Q must be satisfied as follows:

$$C_Q = \int ds\, s\, \Gamma_1^* \left[1 + s\left(\frac{s}{1+s^2} + \frac{\delta}{2}\right)\right](1 - k\delta)$$

$$= \int ds\, s\, \Gamma_2^* \left[1 + s(i_2 + \delta)\right](1 - k\delta)$$

whence:

$$2\lambda \int ds\, \Gamma_2^* s = 2\lambda \int ds\, s\, \Gamma_1^*$$

where again quantities of higher order than Γ have been omitted.

But

$$2C_{E,1} = 2\lambda \int ds\, \Gamma_1^* s + \int ds\, \Gamma_1^* \delta (1 + s^2)(1 - k\delta)$$

which leads again to

$$C_{E,1} = C_{E,2} - C_{E,1-2}$$

The following more general theorem can now be stated.

Given a profile drag angle distribution $\delta = \delta(s)$ the minimum energy loss of a screw propeller is obtained by such a distribution of circulation along the blades that an induced angle distribution i is produced according to the equation:

(51) $$i = \frac{\lambda}{1 - k\delta} \frac{s}{1 + s^2} - \frac{\delta}{2}$$

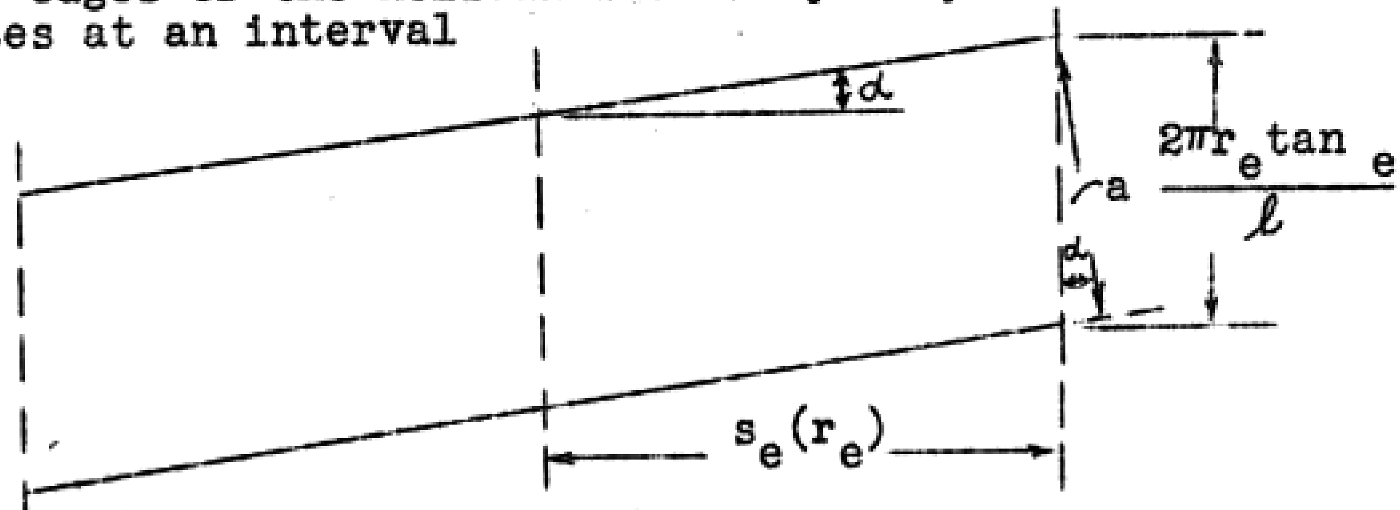
where λ depends on the value of the prescribed torque coefficient C_Q.

Prandtl's Approximation for the Circulation Producing the Induced Angle Change Due to a Rigid Sheet.

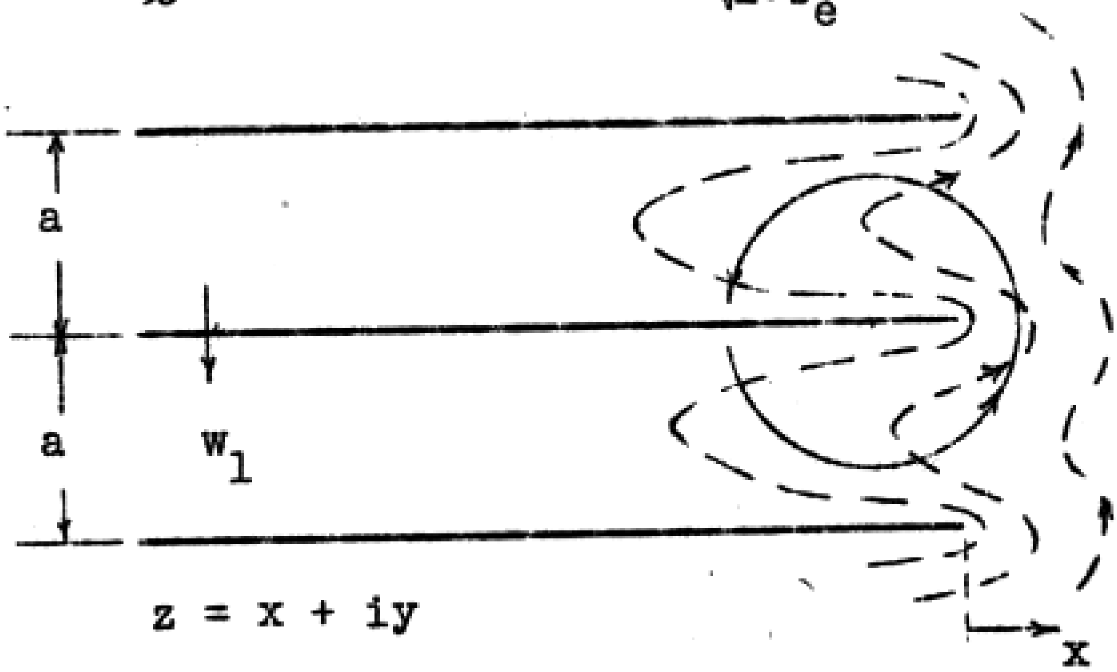
The exact solution of the Laplace equation of the flow produced by a set of helical rigid sheets moving axially or, what gives the same effect rotating about their axis is very complicated as will be seen in the later chapters.

A treatment of the problem has been given by L.Prandtl starting from the fact that the difference between the self-induction $i_m = \frac{\Gamma^*}{4\pi s}$ (See equation(p.23)) and the exact value i can be important only near the blade tips, that is, near the edges of the helical sheets, where the radial escape of the flow on the pressure side of the blade across the tip towards the suction side is most active.

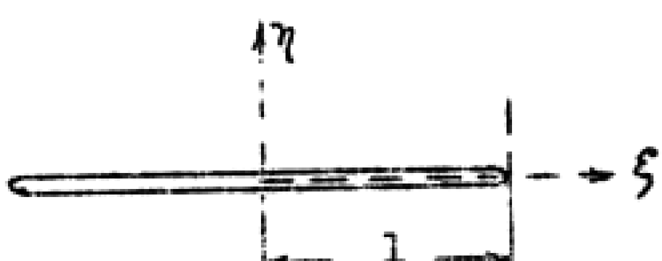
To simplify the problem Prandtl replaces the region near the edges of the helical sheets by a system of half infinite plates at an interval



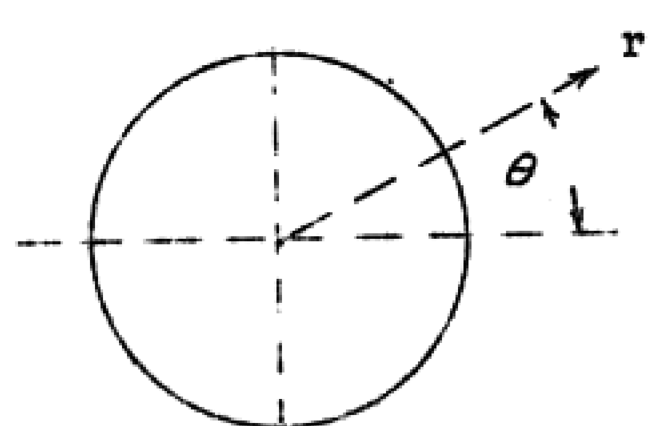
$$a = \frac{2\pi r_e}{\ell}\tan\alpha_e\,\cos\alpha_e = \frac{2\pi w}{\omega\ell}\,\frac{s_e}{\sqrt{1+s_e^2}}$$



$z = x + iy$



$s = \xi + i\eta = \frac{\pi}{a}e^z$



$t = re^{i\theta}$, $s = \frac{1}{2}(t + \frac{1}{t})$

$F = \phi + i\psi = ic \log t = ic \log r - c\theta$

$$\phi = -c\theta \ , \ \psi = c \log r, \quad t = e^{\frac{F}{ic}}$$

$$z = \frac{a}{\pi}\log\left(\frac{1}{2}e^{\frac{F}{ic}} + \frac{1}{2}e^{\frac{-F}{ic}}\right) = \frac{a}{\pi}\log\left(\cos\frac{F}{c}\right)$$

$$F = c \cos^{-1} e^{\frac{\pi z}{a}} = c \cos^{-1}\left(e^{\frac{\pi x}{a}} \ e^{\frac{i\pi y}{a}}\right)$$

and for y = na

$$F = \phi = c \cos^{-1} e^{\frac{\pi x}{a}} , \quad x \leqq 0$$

The potential ϕ has for y = na two equal and opposite values

$$\phi = \pm c \left| \cos^{-1} e^{\frac{\pi x}{a}} \right|$$

The potential therefore at the sheets makes a jump of the amount

$$\Delta \phi = 2c \cos^{-1} e^{\frac{\pi x}{a}}$$

$$= 2c \cos^{-1}\left(e^{-\frac{\pi(r_e - r)}{a}}\right)$$

$$= 2c \cos^{-1}\left(e^{-\frac{\ell}{2}(1-\frac{s}{se})\sqrt{1+s_e^2}}\right) = \Gamma_1$$

Putting $c = \frac{1}{\pi}$ in order to get $\Gamma = 1$ for $s = -\infty$ one arrives at

(52) $$\Gamma = \frac{2}{\pi}\cos^{-1}\left(e^{-\frac{\ell}{2}(1-\frac{s}{se})\sqrt{1+s_e^2}}\right)$$

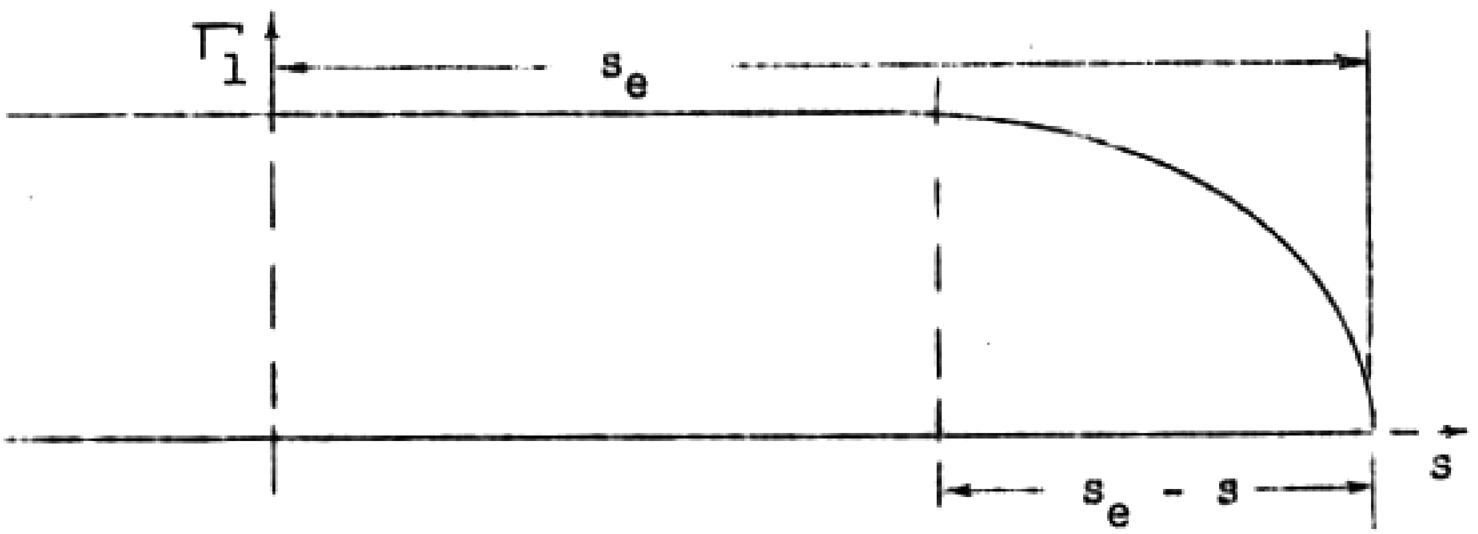


Figure 14.

Prandtl has suggested using this circulation function Γ_1 as a factor with which to multiply the circulation obtained from theories not satisfying the condition of $\Gamma = 0$ at the blade tips.

In fact the product $\Gamma_o \Gamma_1$ with any function Γ_o of the radius furnishes a function which has the desired sharp decrease of the circulation $\Gamma_o \Gamma_1$ towards the tip; this decrease is faster the greater the number ℓ of the blades and the smaller the pitch angle since $(1 + s_e^2)^{-\frac{1}{2}} = \sin\alpha$.

Numerical comparison with the exact theory of Goldstein has given a surprisingly good agreement especially for high values of s_e (small values of α), while for the low values of s_e the approximation of the formula (52) underestimates the decrease of the circulation near the tip.

The Approximate Blade Theory.

Prandtl has suggested using the formula for the distribution K of circulation near the edges of a solidified vortex sheet system normalized to an asymptotic value 1 by multiplying the mean value distribution of the induced angle i_m by K in order to represent the hydrodynamically necessary decrease of the circulation $\Gamma^* = 4\pi s i_m$ to zero at the blade tip, so that

(53) $$i_m = iK(s)$$

where the function K, following the suggestion of Prandtl, is taken as the same as the Γ_1 of (52).

The thrust and torque coefficients change then to

(54) $$C_T = 4\pi\int_{s_i}^{s_e} ds\ s\,Ki(s - (i + \delta))(1 - k\delta)$$

(55) $$C_Q = 4\pi\int_{s_i}^{s_e} ds\ s^2 Ki(1 + s(i + \delta))(1 - k\delta)$$

The proposition of maximum thrust for prescribed torque leads then by a simple application of the isoperimetric method introducing a Lagrangian factor μ to

$$\frac{\partial}{\partial i}\left[i(s - (i + \delta)) - \mu s i(1 + s(i + \delta))\right] = 0$$

and therefore to

(56) $$i + \frac{\delta}{2} = (1 - \mu)\frac{s}{1 + \mu s^2}\,.$$

Observing that $1 - \mu$ must be a small quantity of the order of magnitude i and therefore μ nearly equal to unity one sees that any relation K assumed between i_m and i as long as it does not contain i itself leads back to the general result of a practically rigid vortex sheet as well for the form drag angle $\delta = 0$ as also for a given distribution $\delta = \delta(s)$.

Prandtl's formula in fact does not contain i and his suggestion (53), though it seems somewhat arbitrary, agrees quite well with the results of the rational and in a way exact theory of S. Goldstein which will be given in the next chapter.

Yet if one chooses as the helical vortex sheet not the one formed by undisturbed relative streamlines but the one corresponding to the relative streamlines as they are deflected at the trailing edge of the blade this change has in the expression for K the effect that the distance a between the rigid sheets is changed because the pitch angle α is changed so that

$$s_e = \cot\alpha_e = \frac{\omega r - v^*}{w + w^*} = s_{e,o}\,\frac{1 - \frac{i}{s_{e,o}}}{1 + i s_{e,o}}$$

and, as i can be assumed small relative to $s_{e,o}$ the factor $\sqrt{1 + s_e^2}$ becomes

$$\sqrt{1 + s_e^2} = \sqrt{1 + s_{e,o}^2}(1 - i s_{e,o})$$

so that

(52a) $$K = \frac{2}{\pi}\arccos e^{-\frac{\ell}{2}(1 - i s_e)\sqrt{1 + s_e^2}\left(1 - \frac{s}{s_e}\right)}$$

For this second order approximation in K, in finding the maximum one must allow i to vary also in the factor K in (54) and (55). The expression for the distribution of i and Γ^* corresponding to (p.23) has not yet been used for numerical evaluation of the thrust and torque coefficients. It would cover the case where the inflow angle change is no longer small in comparison to the undisturbed inflow angle α, that is for a propeller in the take off or steep climbing stage of flight.

The further procedure for the first approximation method using (53),(54),(55) and (56) will only be sketched here.

If the inner and outer limits s_i and s_e, that is, the hub radius r_i and the tip radius r_e, are prescribed, then it is sufficient to introduce i from (56) and K from (52) into the integrals for C_T and C_Q and to evaluate them by some graphical method, since they are analytically difficult because of the inconvenient function K. The Lagrangian factor μ must then be determined from the prescribed value of C_Q and then the maximum value of C_T can be numerically evaluated.

But there is also another maximum problem, because not only the best distribution but also the best tip radius r_e or, equivalent with it, the radial number $s_e = \frac{\omega r_e}{w}$ will be desired, except in cases where it is already given by the height of the under carriage or the necessity to avoid air-speeds approaching the velocity of sound.

The differentiation in respect to the upper limit s_e leads to very involved formulae. Here also, therefore, it is better to find the tip radius for maximum thrust by a graphical method. Figure (15) shows how this has been done in a research project worked out by the author; the figure will be self explanatory.

The Potential Problem Solved by S. Goldstein.

Goldstein has considered the Betz theorem for a velocity field induced by a set of helical solidified vortex sheets giving the minimum energy loss affiliated to a screw propeller. He considers in the same way as Prandtl and Betz a set of sheets extending to infinity in both directions $z = \pm\infty$, so that the actual induced velocity when the sheets extend only from $z = 0$ to $z = +\infty$ at the propeller disc are only half of the induced velocities figured for the complete set. For the application later on it is sufficient to know the velocities and circulations at the propeller disc and far behind it.

The $\pm\infty$ doubling of the set of helical sheets reduces the situation to a kind of two dimensional problem. In fact the equation of the screw surfaces are, if w is the velocity of advance of the propeller

$$\theta - \frac{\omega z}{w} = 0, \frac{2\pi}{l}, \frac{4\pi}{l}\ldots$$

$$r < r_e$$

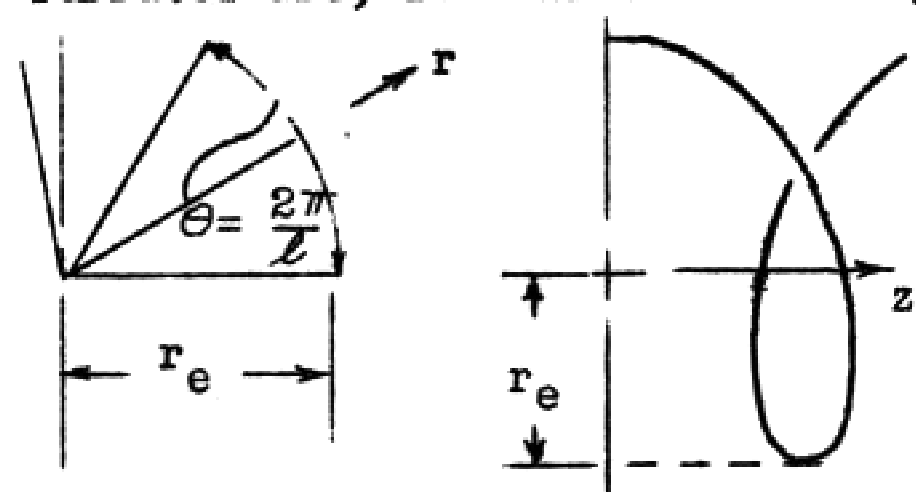


Figure 15.

J

Instead of	read
M	Q
S	T
ρ_a	s_e

$\ell = 2$ $\ell = 3$ $\ell = 4$ $\ell = \infty$

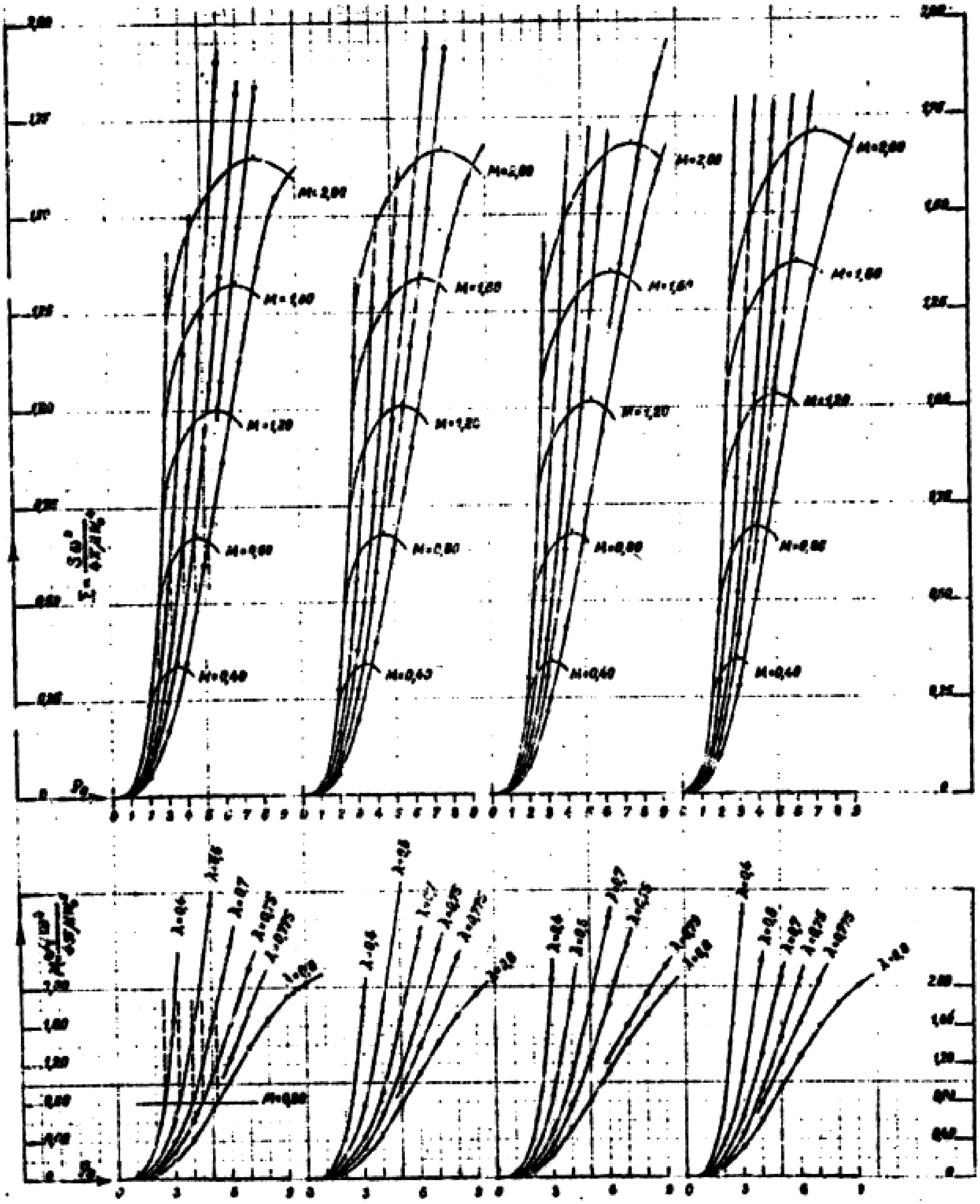


Figure 15.

If again w_1 is the axial velocity of the helical set producing the induced velocities, the boundary conditions

$$w_1 \cos\alpha = \frac{\partial\phi}{\partial z}\cos\alpha - \frac{\partial\phi}{r\partial\theta}\sin\alpha$$

state that the normal component of w_1 shall be equal to the normal component of the fluid velocity at the helical surface.

This can also be written $(\cot\alpha = \frac{\omega r}{w})$

(57) $$w_1\omega r = \frac{\partial\phi}{\partial z}\omega r - w\frac{\partial\phi}{r\partial\theta}$$

for $\theta - \omega z = 0, \frac{2\pi}{\ell}, \frac{4\pi}{\ell}\ldots$ and $r_i < r < r_e$.

The helical symmetry shows that the velocity components $\frac{\partial\phi}{\partial z}$, $\frac{\partial\phi}{r\partial\theta}$ and $\frac{\partial\phi}{\partial r}$, and consequently also the potential ϕ itself, from consideration of continuity inside and outside the propeller jet and from the required zero velocity for $r = \infty$, can be functions only of r and

(52) $$\theta - \frac{\omega z}{w} \equiv \zeta \quad ,$$

so that $$\frac{\partial\phi}{\partial z} = -\frac{\partial\phi}{\partial\zeta}\frac{\omega}{w}, \quad \frac{\partial\phi}{\partial\theta} = \frac{\partial\phi}{\partial\zeta}$$

and the boundary condition becomes by again introducing the notation $\frac{\omega r}{w} = s$

(57a) $$\frac{\partial\phi}{\partial\zeta} = -\frac{s^2}{1+s^2}\frac{w_1 w}{\omega}$$

for $\zeta = 0, \frac{2\pi}{\ell}, \frac{4\pi}{\ell}\ldots$ and $r_i < r < r_e$.

The differential equation $\nabla^2\phi = 0$ can then be written

(58) $$s\frac{\partial}{\partial s}\left(s\frac{\partial\phi}{\partial s}\right) + (1+s^2)\frac{\partial^2\phi}{\partial\zeta^2} = 0$$

Goldstein puts temporarily $\frac{w_1 w}{\omega} = 1$, that is he introduces temporarily a potential function

(59) $$\phi_1 = \phi \frac{w_1 w}{\omega}$$

so that the boundary condition for the inner field $r < r_e$ becomes

(57b) $$\frac{\partial \phi_1}{\partial \zeta} = - \frac{s}{1 + s^2} \quad \text{for} \quad \zeta = 0 \quad \zeta = \pi : \ell = 2,$$

and the differential equation in ϕ_1 remains the same as in ϕ. In addition, ϕ_1 must be single valued between the helical sheets, its derivatives must vanish when $r = \infty$ and it must be continuous except at the screw surfaces. Only the outline of the analysis and some of the results will be given here. (For the details we refer to: On the Vortex Theory of Screw Propellers, by Sidney Goldstein, Proceedings Royal Society of London, Series A Vol. CXXIII, 1929 pp. 440 - 465.)

Consider first the region outside the radius r_e. In this region the boundary condition (57b) does not hold and ϕ_1 must be a continuous function which can be expanded in a series of sines of even multiples of ζ; substituting $\phi_1 = f(s)\sin(2n\zeta)$, $r > r_e$ into (58) one arrives at a set of Bessel equations

(58a) $$s \frac{\partial}{\partial s}\left(s \frac{\partial f_i}{\partial s}\right) - 4n^2(1 + s^2)f = 0$$

so that the complete solution outside r_e is

$$\phi_1 = \sum_{n=1}^{\infty} \sin(2n\zeta) \left[AI_{2n}(2ns) + BK_{2n}(2ns) \right]$$

But the condition that the velocities vanish at $s = \infty$ makes $A = 0$ and the solution can then be written

(60) $$\phi_1 = \sum_{n=1}^{\infty} c_n \sin(2n\zeta) \frac{K_{2n}(2ns)}{K_{2n}(2ns_e)}, \quad B = \frac{e_n}{K_{2n}(2ns_e)}$$

so that the constants c_n are the Fourier constants of the development of ϕ_1 on the circle $s = s_e$, which implies the assumption that the expansion holds also for $s = s_e$ and that the constants c_n can be determined by the condition of continuous transition of this solution for the outer field into the solution for the inner field developed in the next chapter

Proceeding then to the inner region $r < r_e$ between the rigid helical sheets the differential equation (58) must again be satisfied but with the boundary condition (57b) and besides with the condition of continuous transition of the solution to the solution (60) in the outer field.

The integration of (58) fulfilling (57b) leads after some transformations to

$$(61) \quad \phi_1 = \sum_{m=0}^{\infty} \left[\frac{4}{\pi} \frac{T_{1,2m+1}((2m+1)s)}{(2m+1)^2} + a_m \frac{I_{2m+1}((2m+1)s)}{I_{2m+1}((2m+1)s_e)} \right] \cos(2m+1)\zeta$$

In this expression the constants a_m must be determined by the condition of continuous transition and $T_{1,\nu}(z)$ is defined as a Lommel function in which the term containing the modified Bessel function $K_\nu(z)$ has been cancelled, by:

$$T_{1,\nu}(z) = \frac{\nu\pi}{2\sin\frac{1}{2}\nu\pi} I_\nu(z) - t_{1,\nu}(z)$$

and $t_{1,\nu}(z)$ is one of the functions composing a Lommel function namely

$$t_{1,\nu}(z) = m \;_{1} \; \frac{z^{2m}}{(2^2 - \nu^2)(4^2 - \nu^2)\ldots(4m^2 - \nu^2)}$$

a comparison of (61) with (60) requires the development of $\cos(2m+1)\zeta$ in a Fourier series in $\sin 2n\zeta$ and leads after the elimination of the c_n to an infinite number of equations for the a_m, which are solved approximately by starting from the solution of the plane flow between rigid straight sheets (laminae) of the same edge radius r_e.

Observing finally that $\phi = \phi_1 \frac{\omega}{w_1 w}$ and that at the vortex sheet boundaries the jump of the potential is equal to the circulation, Goldstein derives from (61) the following expression for the distribution of the circulation Γ along the radius:

$$(62)\quad \frac{\Gamma\omega}{\pi w_1 w} = G(s) - \frac{2}{\pi}\sum_{m=0}^{\infty}\left(\frac{s_e^2}{1+s_e^2}A_m - \varepsilon_m\right)\frac{I_{2m+1}((2m+1)s)}{I_{2m+1}((2m+1)s_e)}$$

where
$$G_m(s) = \pi\left[\frac{s^2}{1+s^2} - \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{F_{2m+1}(s)}{(2m+1)^2}\right]$$

$$F_{2m+1}(s) = \frac{s^2}{1+s^2} - T_{1,2m+1}((2m+1)s)$$

$$A_o = 1,\ 3A_1 = \tfrac{1}{2},\ 5A_2 = \frac{1\cdot 3}{2\cdot 4},\ 7A_3 = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,.$$

$$a_m = -\frac{s_e^2}{1+s_e^2}A_m + \varepsilon_m$$

and the small correction values of ε_m are given by

ε \ s_e	2	3	5
ε_o	-0,061	-0,047	-0,033
ε_1	+0.013	+0.007	+0.004

partly interpolated and partly neglected for higher values of s_e. With these analytical, expressions tables are figured and represented in the curves given in the diagrams (16) following.

In comparison Prandtl's approximation formula

$$(63)\qquad \Gamma = \frac{s^2}{1+s^2}K$$

of the preceding chapter is also represented by the dotted curves, which show the same character as Goldstein's curves throughout; the agreement improves with increasing ratio s_e of rotation tip speed ωr_e to advance speed w; the thrust values are too high for high pitch (low values of s_e).

But for high values of s_e, that is small values of inflow angles α (Figure (9)), the assumption of cylindrical helical vortex sheets, that is, of negligible deformation of the vortex sheet by its own induced velocities becomes less and less allowable. The limit could be estimated by figuring the excess of $\phi_2 = \frac{v_2}{v_o}$ in the reaction theory and from it the deformation of the vortex sheet two or three diameters of the propeller disc behind it.

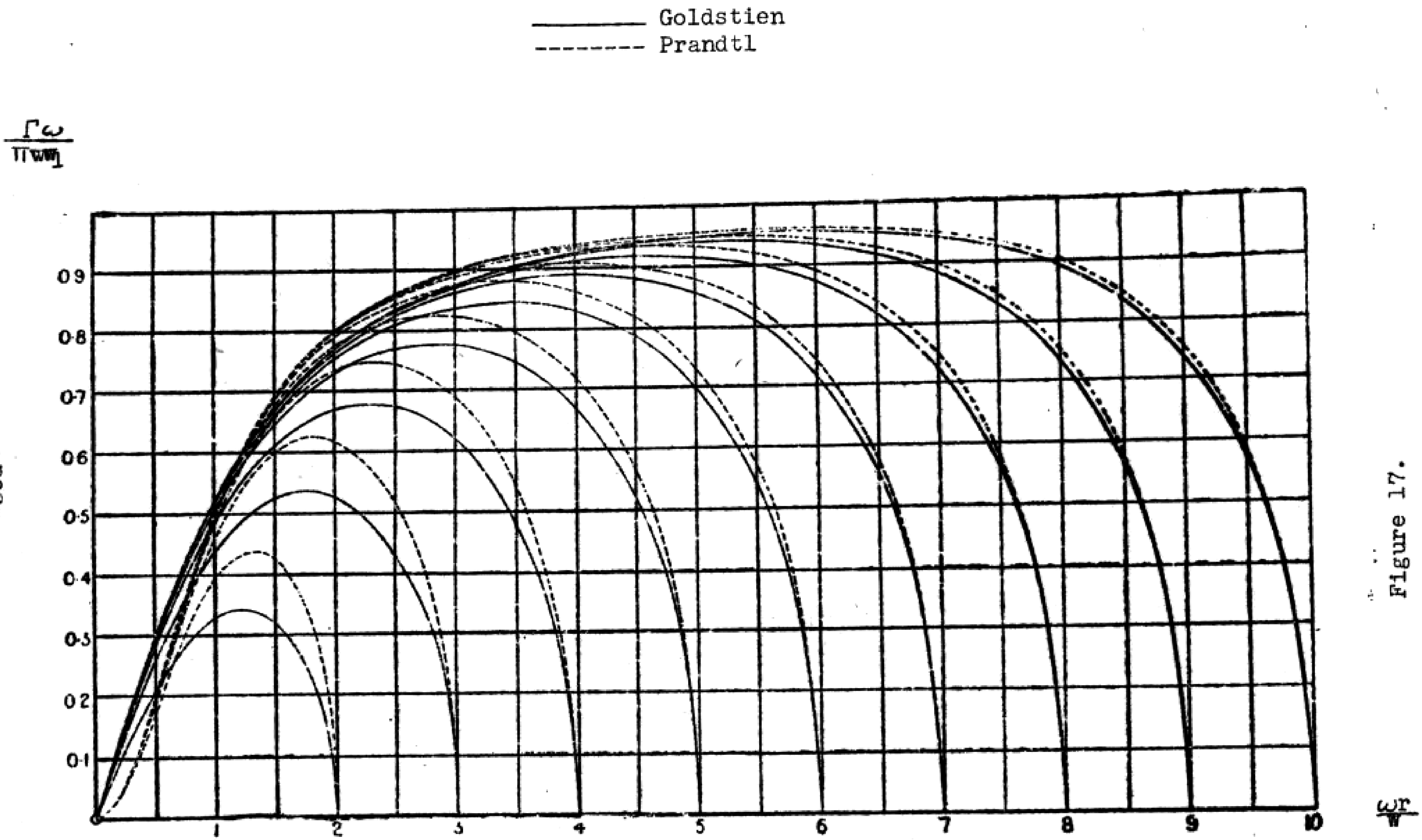


Figure 17.

THE DISTRIBUTION OF CIRCULATION ALONG A PROPELLER BLADE. FOR A TWO-BLADED PROPELLER, WHEN THE ENERGY LOST IN THE SLIP STREAM IS A MINIMUM FOR A GIVEN THRUST.

Inflow Angles Induced by Given Vortex Distributions.

While the formula of Prandtl and the more exact theory of S.Goldstein aim at the determination of the vortex distribution corresponding to the special induced angle distribution of a set of rigid helical sheets, the author has given a method of solving the inverse problem of determining the angles (and velocities) induced by given vortex distributions. In principle this problem is solvable by the application of the Biot-Savart induction law of the effect of all the vortex elements on a point of observation and in the last chapter an application of this principle by Troller will be discussed. Yet as this method does not make use of the helical symmetry of the problem and as it leads to geometrically very inconvenient integrals and to certain difficulties at the sources of self induction another method has been developed.

It assumes at first a simple linear source distribution in the sector spaces between the vortex sheets giving the helical vortex sheets of arbitrarily prescribed vortex distribution and then removes the source distributions but not the vortex sheets by a series of particular integrals of the Laplace - Poisson differential equation taking account of the source distribution by means of the variation of parameters.

We start with the Laplace - Poisson equation

$$(64) \quad \nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = -q(r,\theta,z).$$

Using again the helical symmetry of cylindrical helical sheets going from $z = -\infty$ to $z = +\infty$ with the "helical" variable

$$\theta - \omega z = \zeta$$

and the radial variable $\frac{\omega r}{w} = s$ as before (64) changes into

$$(65) \quad \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\phi}{\partial s}\right) + \frac{\partial^2\phi}{\partial\zeta^2}\left(1 + \frac{1}{s^2}\right) = -\frac{w^2}{\omega^2}\,q(s_1\zeta).$$

First part of the solution.

We assume a source distribution (later on to be removed) given by an arbitrary function odd in ζ, $\zeta = 0$ being the radius halfway between the vortex sheets. For simplicity we choose a function linear in ζ.

$$\frac{w^2}{\omega^2}q(s_1\zeta) = -\zeta p(s), \qquad -\frac{\pi}{\ell} < \zeta < +\frac{\pi}{\ell}$$

repeating itself for each sector space.

The potential function following from this assumption as solution of (65) is

$$\phi_o = \zeta\left[c_1 \log s + c_2 + \int_{s_i}^{s} \frac{d\xi}{\xi} \int_{s_i}^{\xi} d\psi \; \psi \; p(\psi)\right]$$

and with the condition that ϕ_o and its derivatives be finite at the axis and zero at infinity

$$\phi_o = \int_{s_i}^{s} \frac{d\xi}{\xi} \int_{r_i}^{\xi} d\psi \, \psi \, p(\psi) \; .$$

The velocities derived from this potential are

(66)

$$u^{(o)} = \frac{\partial \phi_o}{\partial r} = \frac{\omega}{w} \zeta \, s^{-1} \int_{s_i}^{s} d\psi \, \psi p(\psi)$$

$$v^{(o)} = \frac{\partial \phi_o}{\partial \theta} = \frac{\omega}{w} s^{-1} \int_{s_i}^{s} \frac{d\xi}{\xi} \int_{o}^{\xi} d\psi \, \psi \, p(\psi)$$

$$w^{(o)} = \frac{\partial \phi_o}{\partial z} = \frac{\omega}{w} \int_{s_i}^{s} \frac{d\xi}{\xi} \int_{o}^{\xi} d\psi \, \psi \, p(\psi)$$

and only ϕ^o and the radial velocity component change discontinuously at the radii

$$-\frac{\pi}{\ell}, +\frac{\pi}{\ell}, \frac{3\pi}{\ell} \text{ — } \frac{(2\ell - 3)\pi}{\ell}$$

while the circumferential velocities $v^{(o)}$ and the axial velocities $w^{(o)}$ are continuous across these radii, being constant along each circumference r; also the pressure is continuous.

The circulation Γ being equal to the potential discontinuity is given by

(67) $$\Gamma = (2\,\phi_o)_{\xi=\frac{\pi}{\ell}} = \frac{2\pi}{\ell} \int_{o}^{s} \frac{d\xi}{\xi} \int_{o}^{\xi} d\psi \, \psi \, p(\psi)$$

and the vortex strength

(68) $$E = \frac{d\Gamma}{dr} = \frac{2\pi}{\ell} \frac{\omega}{w} s^{-1} \int_{o}^{s} d\psi \, \psi \, p(\psi) = 2\,u^{(o)}$$

where vortex strength and discontinuity of radial velocity are equal, as is necessary.

The inversion of this last equation is especially simple for the source function linear in ζ namely

$$p = \frac{\ell}{2\pi} \frac{1}{s} \frac{d}{ds}\left(s \frac{d\Gamma}{ds}\right) \tag{68a}$$

so that the source distribution can be adapted to any chosen vortex distribution.

Second part of the solution.

Since the assumed source distribution is not actually existent, it must be removed by a second additional solution of the Laplace - Poisson equation with the opposite source distribution but without discontinuity of potential and radial velocity.

We take a trigonometric series running over the whole circumference in the periodic form

$$\phi_1 = \sum_{n=1}^{\infty} R_n \sin(n\ell\zeta) \tag{69}$$

in which every term assumes the value zero for $\zeta=0, \zeta=\frac{2\pi}{\ell}\ldots$ on the halfway radii and on the vortex sheets themselves, so as not to change the potential values ϕ_o at the vortex sheets and to have the proper period and symmetry of the set up.

If one uses the convenient dimensionless radial variable

$$n\ell s = \sigma_n$$

(65) takes the form

$$\frac{1}{\sigma_n}\frac{d}{d\sigma_n}\left(\sigma_n \frac{dR_n}{d\sigma_n}\right) - R_n\left(1+\left(\frac{n\ell}{\sigma_n}\right)^2\right) = -\frac{\ell}{\pi(n\ell)^2} p \int_{-\frac{\pi}{\ell}}^{+\frac{\pi}{\ell}} \zeta \sin n\ell\zeta \, d\zeta \tag{65a}$$

$$= -(1)^n \frac{2p}{(n\ell)^3}\,.$$

(65a) is the equation of modified Bessel functions and the solution with the integration constants f_n and g_n can be written

$$R_n = (-1)^n \frac{2}{(n\ell)^3}\left[f_n I_{n\ell}(\sigma_n) + g_n K_{n\ell}(\sigma_n) + R_{n,\text{part.}}\right]$$

The particular integral may be determined by variation of parameters as follows:

$$R_n' = f'I + g'K = 0$$

$$R'' = f'I' + g'K' = (-1)^n \frac{2p}{(n\ell)^3}$$

$$f'(I'K - K'I) = (-1)^n \frac{2p}{(n\ell)^3} K$$

$$g'(K'I - I'K) = (-1)^n \frac{2p}{(n\ell)^3} I$$

Now using the general relation

$$I_{n\ell} \frac{dK_{n\ell}(\sigma_n)}{d\sigma_n} - K_{n\ell} \frac{dI_{n\ell}(\sigma_n)}{d\sigma_n} = - \frac{1}{\sigma_n}$$

one obtains

$$f_n' = -(-1)^n \frac{2p}{(n\ell)^3} K \sigma_n$$

$$g_n' = +(-1)^n \frac{2p}{(n\ell)^3} I \sigma_n$$

and with these values the complete integral:

$$R_n = (-1)^n \frac{2p}{(n\ell)^3} \Big[f_n I_{n\ell}(\sigma_n) + g_n K_{n\ell}(\sigma_n) + \int_{\sigma_{n,i}}^{\sigma_{n,e}} d\tau_n \cdot \tau_n \cdot p(\tau_n) \left(I_{n\ell}(\sigma_n) K_{n\ell}(\tau_n) - K_{n\ell}(\sigma_n) I_{n\ell}(\tau_n) \right) \Big] \tag{66}$$

From (66) follow the additional velocities, which pass continuously through the helical sheets.

$$\left\{ \begin{aligned} u^{(1)} &= \frac{\omega}{w} \sum \frac{dR_n}{d\sigma_n} n\ell \sin(n\ell\zeta) \\ v^{(1)} &= \frac{\omega}{w} \sum \sigma_n^{-1} (n\ell)^2 R_n \cos(n\ell\zeta) \\ w^{(1)} &= - \frac{\omega}{w} \sum n\ell R_n \cos(n\ell\zeta) \end{aligned} \right. \tag{67}$$

The boundary conditions to be satisfied are that all three velocities must be zero at infinity, and the radial velocity zero at the hub radius ordinate $\sigma_{n,i}$. The condition that $u^{(1)}$, $v^{(1)}$ and $w^{(1)}$ vanish at infinity makes

$$I_{n\ell}(\sigma_n)\left[f_n + \int_{\sigma_i}^{\sigma_e} \tau_n d\tau_n K_{n\ell}(\tau_n) p\right] = 0$$

and so

$$f_n = -\int_{\sigma_i}^{\sigma_e} \tau_n d\tau_n K_{n\ell}(\tau_n) p \ .$$

The condition that $u^{(1)}$ vanishes at the hub radius $\sigma = \sigma_i$ shows that $\left(\frac{dR_n}{d\sigma_n}\right)_{\sigma_n=\sigma_{n,i}} = 0$ and following from it

$$g_n = 0$$

so that

$$\text{(67a)} \quad R_n = -(-1)^n \frac{2}{(n\ell)^3}\left[K_{n\ell}(\sigma_n)\int_{\sigma_i} \tau_n d\tau_n I_{n\ell}(\tau_n) p + I_{n\ell}(\sigma_n)\int_{\sigma}^{\sigma_e} \tau_n d\tau_n K_{n\ell}(\tau_n) p\right]$$

What is needed for the following is the induced velocity component V* perpendicular to the undisturbed velocity V at the radii $\zeta = \frac{\pi}{\ell}, \frac{3\pi}{\ell}, \ldots, \frac{(2\ell-1)\pi}{\ell}$.

This induced velocity must be composed of half the values of the induced velocity components v_1 and w_1 for the reason already explained in the preceding chapters.

Thus we get $$\frac{V^*}{V} = i = \tfrac{1}{2}\left[\frac{w_1}{V}\frac{\omega r}{V} - \frac{w_1}{V}\frac{w}{V}\right]$$

and as w in view of the positive sense of z is negative

$$i = \frac{1}{2w^2(1+s^2)}\left[\left\{w^{(0)} + w^{(1)}\right\}s - \left(v^{(0)} + v^{(1)}\right)\right].$$

Using now formulae (66) and (68) with the angles $\zeta = \frac{\pi}{\ell}, \frac{3\pi}{\ell}$ etc. one finds the expression

$$i = \frac{\omega}{2w^2 s}\left[R_o + \sum_{n=1}^{\infty}(-1)^n\, n\ell\, R_n\right]$$

and inserting the expressions for R_o and p given above, for R_n from (67a) and for $\Gamma = \Gamma^* \frac{w^2}{\ell\omega}$ one arrives finally at:

$$(70)\ i = \frac{\Gamma^*}{4\pi s} - \frac{1}{2\pi s}\left[\sum_{n=1}^{\infty} K_{n\ell}(\sigma_n)\int_{\sigma_{n,i}}^{\sigma_n} d(\tau_n \frac{d\Gamma^*}{d\tau_n}) I_{n\ell}(\tau_n) + I_{n\ell}(\sigma_n)\int_{\sigma_n}^{\sigma_{n,e}} d(\tau_n \frac{d\Gamma^*}{d\tau_n}) K_{n\ell}(\tau_n)\right]$$

It may be observed that

$$\frac{\Gamma^*}{4\pi s} = i_m$$

was found as the average induced angle (taken over the circumference), furthermore that (70) represents the induction by the vortex sheet $\frac{d\Gamma^*}{d\tau}$ as the summation of the effects from the inside of the point of observation $\sigma_n = n\ell s = n\ell \frac{\omega r}{w}$ by the first integral and from the outside of σ_n by the second integral.

The derivation of the velocities induced by a cylindrical system of vortex sheets at the propeller disc section has also been treated by means of the Biot - Savart theorem. However this theorem requires the calculation of integrals depending on all the radii from any point of the vortex sheets to all other points of the circular cross section and further requires a special discussion of the self induction of the points of the vortex sheets where the denominator of the integrand becomes zero. Betz has used these integrals to prove his theorem, Lamb has applied it partly for the theory of the solenoid and Troller has performed some very instructive computations.

We close our discussions by a remark about the relation between the distribution of circulation $\Gamma(s)$ and the design characteristics of the propeller blade consisting of blade width,

$$c = \frac{2\pi r b}{\ell} \text{ (p.21)}$$

of blade pitch angle (Fig. 9), lift coefficient C_L and the undisturbed inflow velocity $V_o = \sqrt{w^2 + \omega^2 r^2}$.

In fact the formula (43) p. 22 derived by comparison of the lift coefficient expression with the Kutta - Joukowski theorem states this relation by

$$C_{L,o}c = \frac{2\Gamma}{V_o}$$

which gives the product of blade shape and pitch angle $\beta = \frac{C_{L,o}}{2\pi}$

$$\beta c = \frac{4\pi\Gamma}{V_o}$$

In Fig. (44) the results of this formula in connection with the results for different blade numbers $\ell = 2,3,4,\infty$, given by the curves on Fig. (44) are shown and it appears that the blade shape corresponds closely to the shapes which have proved efficient in practice.

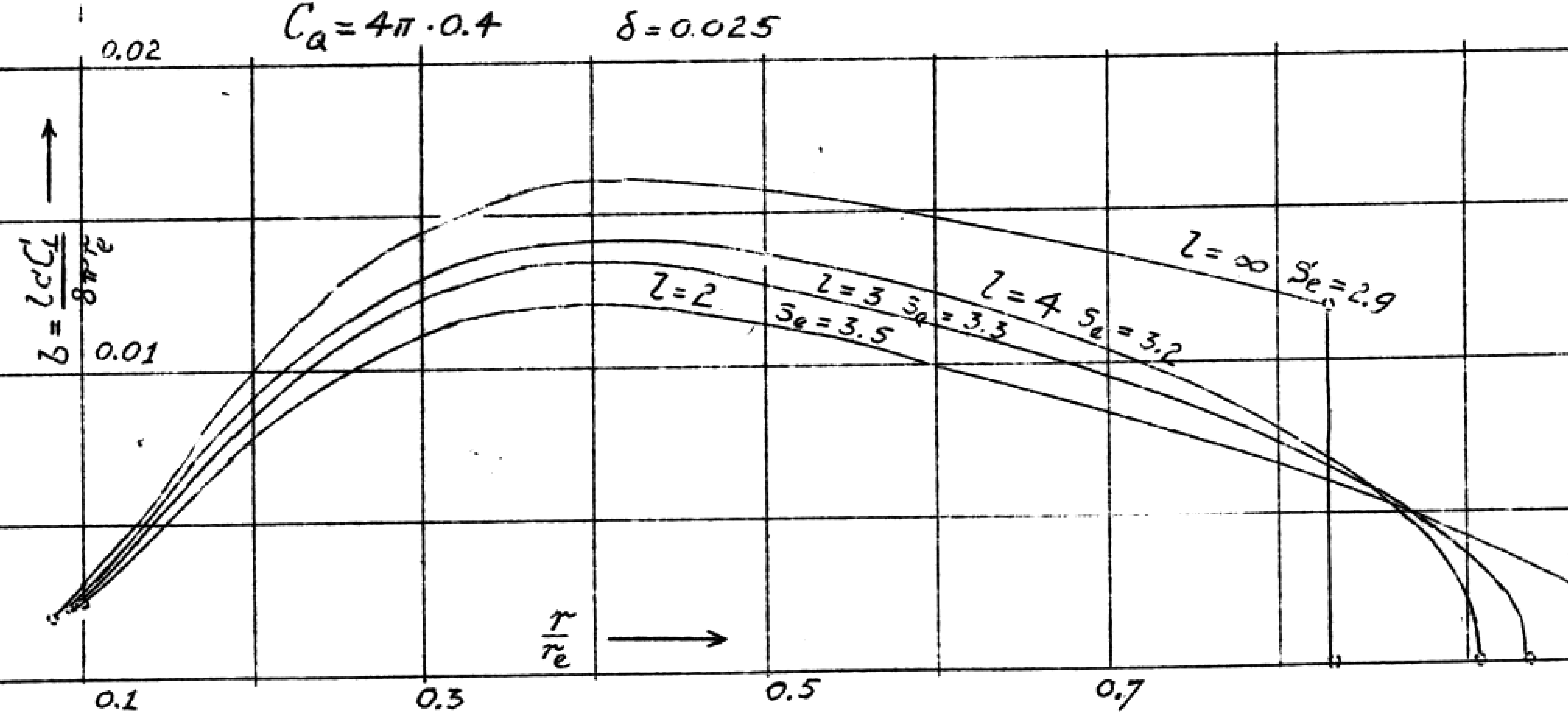
Summary and Literary Remarks

The first part of this course on the hydrodynamics of propellers brings the general properties of any hydrodynamic propelling mechanism following from the application of the general theorems of momentum, moment of momentum and energy on the generation of a jet.(1) This kind of approach furnishes the maximally obtainable effects, but it gives very few indications about the design of propelling mechanisms. However it embraces more than the Blade (2) and Blade - Vortex (3) Theory of the screw propeller in so far as it gives results about the stationary propelling, wind motor and fan mechanisms where mathematical difficulties of the Blade - Vortex Theory have not yet been surmounted.

The second part treats this Vortex Theory of the propeller by a rational application of (2) the circulation - vortex theorems, (1) the lift and drag coefficients of airfoils and the dynamic principles of momentum, moment of momentum and energy.

In particular it starts with the expressions of the inflow velocity and inflow angle change as "induced" by the helical vortex sheets left behind the trailing edges of the advancing and rotating blades. It formulates in a non dimensional way the thrust - torque - and energy loss integrals.
The Theorem of Minimum Induced Energy Loss of the rigid vortex sheet (3) and its generalization including the form drag coefficients is derived by potential theory.(4) It follows then the derivation of an approximative formula for the vortex distribution corresponding to the rigidly induced inflow angle (3) and the "exact" theory of this distribution still only approximately correct for small ratios of induced and undisturbed inflow angle. (5) To indicate a technically practical

Fig. 44



$C_a = 4\pi \cdot 0.4$
$\delta = 0.025$
0.02
0.01
$b = \frac{l c C_L}{8 \pi r_e}$
$\frac{r}{r_e}$
0.1
0.3
0.5
0.7
1
$l = \infty$ $S_e = 2.9$
$l = 4$ $S_e = 3.2$
$l = 3$ $S_e = 3.3$
$l = 2$ $S_e = 3.5$

theory for the best arrangement of thrust distribution and diameter the thrust and torque coefficients are graphically discussed using Prandtl's first named approximative formula.

The inverse approach to determine the induced inflow from given vortex distributions that is from given thrust and torque distributions along the blade is also treated somewhat summarily by the Biot - Savart integral theorem (3)(7) and more in detail by the method of superposition of a simple discontinuous and a continuous solution of certain source distributions. This part closes with a remark about the design characteristics following from distribution of circulation and induced angle.

(1) Rankine, Transactions of the Instit. of Nav. Architects London, 1865
Greenhill, Transactions of the Instit. of Nav. Architects London, 1888

(2) W. Froude, Transactions of the Instit. of Nav. Architects London, 1878
Drzewiecki, Congrès Internat. d. Architectes et d. Constructeurs Navals, Paris 1900

(3) A. Betz and L. Prandtl, Schraubenpropeller mit geringstem Energisverlust, Gottingen, 1927
See also, L. Prandtl, Applications of Modern Aerodynamics to Aeronautics, Report 116, Nat. Advis. Comm. f. Aeronautics.

(4) H. Reissner, A Generalized Vortex Theory of the Screw Propeller and its Application. Technical Note, Advis. Comm. for Aeron. 750, 1940

(5) S. Goldstein, On the Vortex Theory of Screw Propellers, Proc. Royal Soc. of London, Ser. A. Vol. CXXIII 1929 pp. 440 - 465

(6) Helmbold, Z.F.M., 1924 p. 150 - 170
Bienen u.v. Karman, Z. Ver. Deutsch. Ing. 1924 p.1237
H. Reissner, Unpublished report, 1936, lecture given at Mass. Inst. Techn. Sept. 1937

(7) H. Lamb, the magnetic field of a Helix, Proc. Cambr.Phil. Soc. XXI (1923) pp. 477 - 481

Th. Troller, Z.F, Angew. Math u. Mach. 1928